Portfolio Optimization with Constraints on Tracking Error

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Abstract

This paper explores the risk and return relationship of active portfolios subject to a constraint on tracking error volatility (TEV), which can also be interpreted in terms of Value At Risk (VAR). This is the typical setup for active managers who are given the task of beating a benchmark. The problem with this setup is that the portfolio manager pays no attention to the total portfolio risk, resulting in seriously inefficient portfolios, unless some additional constraints are imposed.

This paper shows that TEV-constrained portfolios are described by an ellipse in the traditional mean-variance plane. This yields a number of new insights. We show that, due to the flat shape of this ellipse, adding a constraint on total portfolio volatility can substantially improve the performance of the active portfolio. This constraint is most beneficial in situations with low values of the admissible TEV, or when the benchmark is relatively inefficient.

In a typical portfolio delegation problem, the investor assigns the management of assets to a portfolio manager who is given the task of beating a benchmark. When outperformance is observed for the active portfolio, the issue is whether the added value is in line with the risks undertaken. This is particularly important with performance fees, which induce an option-like pattern in the compensation of the agent, who may have an incentive to take on more risk to increase the value of the option.[1] To control for this, institutional investors commonly impose a limit on the volatility of the deviation of the active portfolio from the benchmark, also known as Tracking Error Volatility (TEV).

The problem with this setup, however, is that it induces the manager to optimize in an excess return space only, totally ignoring the investor's overall portfolio risk. In an insightful paper, Roll (1992) noted that excess return optimization leads to the unpalatable result that the active portfolio has systematically higher risk than the benchmark and is not optimal. Jorion (2002) examines a sample of enhanced index funds, which are more likely to go through a formal excess return optimization, and finds that such funds have systematically greater risk than the benchmark. Thus, the problem is real.

It is not clear why the industry maintains this widespread emphasis on tracking error risk control.[2] Roll conjectured that diversifying across managers could mitigate the inherent flaw in TEV optimization but, as we will show later, this is not the case.

Instead, this paper investigates whether this problem can be corrected with additional restrictions on the active portfolio without eliminating the usual TEV constraint. Thus, this paper takes the TEV constraint as given, even if this restriction is not optimal, given its widespread practice. The paper derives the constant TEV

[1]This is the case, for instance, with hedge funds, which typically charge a variable fee of 20 percent of profits. For a good introduction to the major issues with performance fees, see Davanzo and Nesbitt (1987), Grinold and Rudd (1987), and Kritzman (1987).

[2]Going one step further, Admati and Pfleiderer (1997) examine the rationale for benchmark-adjusted compensation schemes. They argue that such schemes are generally inconsistent with optimal risk-sharing and do not help in solving potential contracting problems with the portfolio manager.

frontier in the original mean-variance space.

Traditionally, TEV has been checked after the fact, i.e. from the volatility of historical excess returns. More recently, the industry has witnessed the advent of forward-looking measures of risk such Value at Risk (VAR).[3] The essence of VAR is to measure the downside loss for *current* portfolio positions, based on the best risk forecast. With a distributional assumption for portfolio returns, excess returns VAR is equivalent to a forward-looking measure of TEV. Nowadays, VAR limits are increasingly used to ensure that the active portfolio does not stray too much from the benchmark.[4] Furthermore, the VAR methodology is now implemented as "risk budgets," which can be defined as the conversion of optimal mean-variance allocations to VAR assignments for active managers.[5]

The spreading use of VAR systems makes it possible to consider other ex-ante restrictions on the active portfolio. To do this, this paper explores the risk and return relationship of active portfolios subject to a TEV constraint. I show that the TEV constraint is described by an ellipse in the usual mean-variance space. This yields several new insights.

For instance, we can explore the effect of adding a constraint that the total portfolio risk be equal to that of the benchmark. Such constraint can be imposed easily using a VAR system, along with the usual TEV constraint. The question is whether the addition of this constraint creates too large a penalty in terms of expected returns. I show that the penalty depends on the "slope" of the constant TEV ellipse and explore the driving factors behind this slope.

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[3]See Jorion (2000) for a detailed analysis of VAR.

[4]Another issue is who should be given authority to control tracking risk. Possible candidates are the investment manager, the plan sponsor, the custodian, or outside consultants. One could argue that risk should be controlled by the portfolio manager. After all, he or she should already have in place a risk measurement system, which gives the tracking error of the active portfolio. The manager should also have the best understanding of the instruments in the portfolio. Thomas (2000), however, argues that there is a conflict of interest for the manager to control VAR and that this function is best performed by a disinterested party.

[5]For an introduction to risk budgeting, see Chow and Kritzman (2001). Lucas and Klaasen (1998) also discuss the link between portfolio optimization and VAR.

The primary contribution of this paper is the derivation and interpretation of these analytical results. Apart from Roll's seminal paper, there are only few publications on this important, and practical, topic.[6] The implications of these analytical results are illustrated with an example.

This paper is structured as follows. Section 1 reviews optimization results for the efficient frontiers in absolute and relative spaces. The main result is that TEV optimization creates an increase in the fund's risk. Section 2 then shows that diversifying across active managers is not likely to reduce much this effect. Next, Section 3 describes the relationship between absolute risk and return of portfolios subject to this TEV constraint. Based on these results, Section 4 then discusses how imposing additional restrictions on the active manager could bring the portfolio closer to the efficient frontier. Finally, Section 5 provides some concluding comments.

[6]The closest paper is that of Leibowitz, Kogelman, and Bader (1992), who discuss the application of the shortfall approach to portfolio choice for a pension fund. In this case, the tracking error volatility is replaced by the "surplus return," defined relative to the liabilities. Their paper, however, involves another constraint, which is a linear relationship between expected returns and volatility and has a simple setup with two risky assets only. In addition, no closed-form solutions are presented. Chow (1995) argues that the objective function should account for total risk but also tracking error risk. Rudolf, Wolter, and Zimmermann (1999) compare various linear models to minimize tracking error. Ammann and Zimmermann (2001) examine the relationship between limits on TEV and deviations from benchmark weights.

1 Efficient Frontiers in Absolute and Relative Space

1.1 Setup

Consider a portfolio manager who is given the task of beating an index, or "benchmark". This involves taking positions in the assets within the index and perhaps other assets. Define the following notations:

$$
\begin{aligned}
q &= \text{vector of index weights for the sample of } N \text{ assets} \\
x &= \text{vector of deviations from index} \\
q_P = q + x &= \text{vector of portfolio weights} \\
E &= \text{vector of expected returns} \\
V &= \text{covariance matrix for asset returns}
\end{aligned}
$$

To preserve linearity, I assume that net short-sales are allowed, i.e. the total active weight $q_i + x_i$ can be negative for every asset i. Otherwise, the problem generalizes to a quadratic optimization for which there is no closed-form solution.

In practice, the benchmark has positive weights q_i. More generally, it can have negative or zero weights on the assets. Thus, the universe of assets can exceed the components of the index. The optimization, however, must include the assets in the benchmark.

We can write expected returns and variances in matrix notation as:

$$
\begin{aligned}
\mu_B = q'E &= \text{expected return on index} \\
\sigma_B^2 = q'Vq &= \text{variance of index return} \\
\mu_\epsilon = x'E &= \text{expected excess return} \\
\sigma_\epsilon^2 = T = x'Vx &= \text{variance of tracking error}
\end{aligned}
$$

Note that these are forward-looking measures, as x represents current deviations, and V the best guess of the covariance matrix over the horizon. Given the initial portfolio value of W_0, the tracking error VAR is:

$$VAR = W_0 \times \alpha \times \sigma_\epsilon. \tag{1}$$

Assuming normally distributed returns, for example, α is set at 1.65 for a one-tailed confidence level of 95%.

The active portfolio expected return and variance are

$$\mu_P = (q+x)'E = \mu_B + \mu_\epsilon \tag{2}$$

$$\sigma_P^2 = (q+x)'V(q+x) = \sigma_B^2 + 2q'Vx + x'Vx \tag{3}$$

The investment problem is subject to a constraint that the portfolio be fully invested, or that the total portfolio weights $(q+x)$ must add up to unity. This can be written as $(q+x)'1 = 1$, with 1 representing a vector of ones. Since the benchmark weights also add up to unity, the portfolio deviations must add up to zero, or $x'1 = 0$. Thus, the active portfolio can be constructed as a position in the index plus a "hedge fund", with positive and negative positions that represent active views.

1.2 The Efficient Frontier in Absolute Return Space

Appendix A reviews the traditional analysis of the Mean-Variance efficient frontier when there is no risk-free asset. The portfolio allocation problem can be setup as a minimization of σ_P^2 subject to a target expected return $\mu_P = G$ and full investment $q_P'1 = 1$ constraint. The solution is given by Equation (12) in Appendix A. The efficient set can be described by a hyperbola in the (σ, μ) space, with asymptotes having a slope of $\pm\sqrt{d}$, where d is a function of the efficient set characteristics. This slope represents the best return-to-risk ratio for this set of assets.

1.3 The Efficient Frontier in Excess Return Space

We now turn to the optimization problem in the excess return space. We can trace out the tracking error frontier by maximizing the expected excess return $\mu_\epsilon = x'E$ subject to a constraint on tracking error variance $x'Vx = T$ and $x'1 = 0$. The solution, reviewed in Appendix B, is

$$x = \pm\sqrt{\frac{T}{d}}\, V^{-1}[E - \mu_{MV}1] \tag{4}$$

where μ_{MV} is the expected return of the global minimum-variance portfolio. Roll (1992) notes that this solution is totally independent of the benchmark, as it does not involve q. This yields the unexpected result that active managers pay no attention to the benchmark.[7] This is a far-reaching result, as this behavior is not optimal for the investor.

In a mean-volatility space in excess returns, the (upper) efficient frontier is

$$\mu_\epsilon = \sqrt{d}\sqrt{T} = \sqrt{d}\ \sigma_\epsilon \tag{5}$$

which is linear in the tracking error volatility $TEV = \sqrt{T} = \sigma_\epsilon$, as shown in Figure 1.[8] The benchmark is on the vertical axis since it has zero tracking error.

In Equation (5), the coefficient $\sqrt{d}$ also represents the **Information Ratio** (IR), defined as the ratio of the expected excess return over the TEV. The IR is commonly used to compare investment managers. Grinold and Kahn (1995), for example, assert that an IR of 0.50 is "good." I choose the efficient set parameters so that $\sqrt{d} = 0.50$.

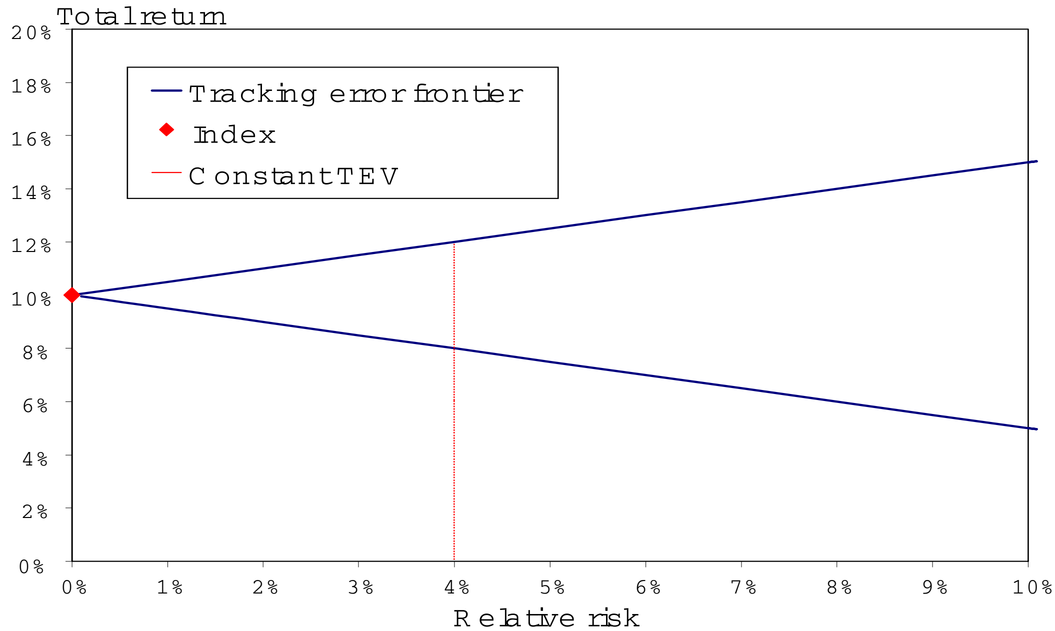
If the manager is solely measured in terms of excess return performance, he or she should pick a point on the upper part of this efficient frontier. For instance, the manager may have a utility function that balances expected value added against tracking error volatility. Note that, since the efficient set consists of a straight line, the maximal Sharpe ratio is not a usable criterion for portfolio allocation.

In practice, expected returns are not observable nor verifiable by the investor. Instead, the portfolio manager is given a constraint on tracking error volatility, which determines the optimal allocation. This is represented by the intersection of the efficient set with the vertical line representing a constant σ_ϵ. Figure 1 shows the case where $\sigma_\epsilon = 4\%$, which is a typical constraint for active managers. With $IR = 0.5$, this translates into an expected excess return of 200 basis points (bp).

[7] In practice, the active positions will depend on the benchmark if there are short-selling restrictions on total weights. Assets with low expected returns can only be shorted up to the extent of the (long) position in the benchmark.

[8] With restrictions that the total portfolio weights cannot be negative, $q_i + x_i \geq 0$, the efficient frontier starts as a straight line, then becomes concave as some of the restrictions become binding $x_i = -q_i$. It then flattens out until the whole active portfolio is invested in the asset with the highest expected return.

FIGURE 1 Tracking Error Frontier in Excess Return Space



1.4 The TE Frontier in Absolute Return Space

Based on this information, we can trace the TE frontier in the traditional absolute return space, as Roll (1992) did. Figure 2 displays this frontier as a line with markings, going through the benchmark. Away from the index, each mark represents a fixed value for the tracking error volatility, e.g. 1%, 2%, and so on.

The graph shows an unintended effect of tracking error optimization: Instead of moving toward the true efficient frontier, i.e. up and to the left of the index, the tracking error frontier moves up and to the right. This increases the total volatility of the portfolio, which is a direct result of focusing myopically on excess returns instead of total returns.

FIGURE 2 Efficient Frontier in Absolute Return Space

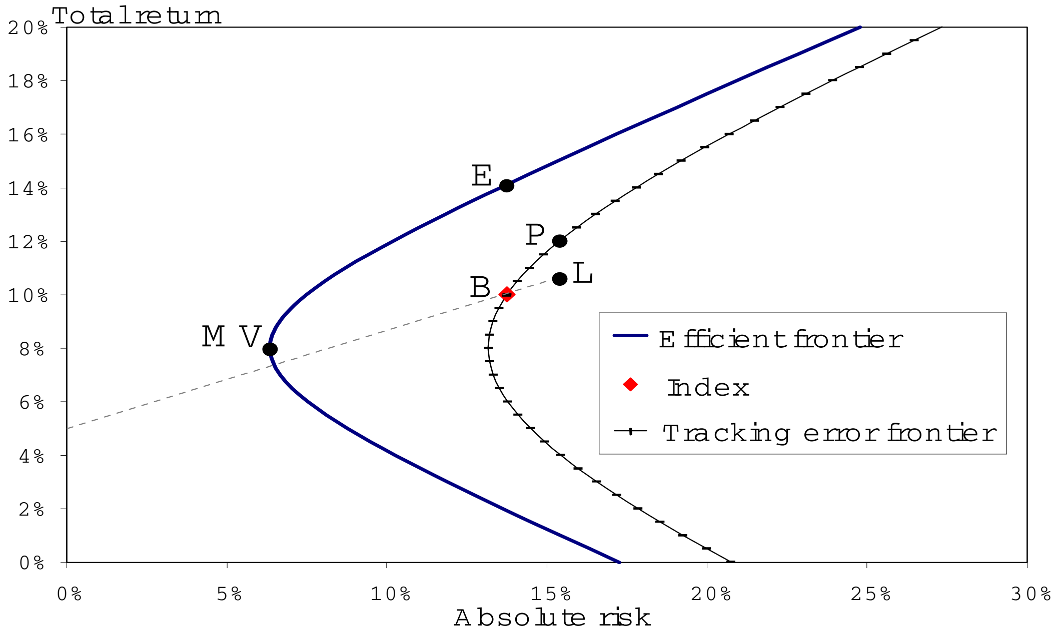


Table 1 displays the characteristics of the efficient frontier and the index for this data set. The index B has an expected return of 10%, with a volatility of 13.8%, which is typical of a well-diversified global equity benchmark. With a 5% risk-free rate, its Sharpe ratio is 0.36.

In this example, the data are taken from the global equity indices provided by from Morgan Stanley Capital International. Unhedged total returns are measured in dollars over the period 1980-2000 for the U.S., U.K., Japan, and Germany. In addition, we use as a fifth asset the Lehman Brothers U.S. Aggregate Bond Index. The covariance matrix is taken from historical data. Expected returns are arbitrary and chosen so as to satisfy the efficient set parameters.

The minimum variance portfolio MV has expected return of 8.0% with volatility of 6.4%. Its expected return $\mu_{MV} = 8\%$ is less than that of the index $\mu_B = 10\%$, which should be the case—otherwise the index would be grossly inefficient.

The efficient portfolio E is defined as the portfolio on the efficient frontier with the same level of risk as the index. Here, $\mu_E = 14.1\%$. These numbers are typical of the expected performance of active managers, as they are based on an information ratio of $\sqrt{d} = 0.50$.

TABLE 1 Efficient Set and Index Characteristics

Portfolio	μ	σ
Index (B)	10.0%	13.8%
Global Minimum Variance (MV)	8.0%	6.4%
Efficient Portfolio (E)	14.1%	13.8%
4% Tracking Risk Portfolio (P)	12.0%	15.4%
Leveraged Benchmark (L)	10.6%	15.4%

NOTE: The table reports the expected return (μ) and volatility (σ) of various portfolios in percent per annum. Portfolio B is the benchmark, portfolio MV achieves the global minimum variance, portfolio E has the same risk as B but is efficient, portfolio P has 4% tracking error risk, and portfolio L leverages up the benchmark to have the same risk as P.

Let us focus now on the 4% tracking risk portfolio P. This has expected return of 12.0%, with total risk of 15.4%. Part of the 200 bp increase in expected return relative to the benchmark, however, is illusory, as it reflects the higher risk of the portfolio. To see this point, Figure 2 considers a leveraged portfolio L, achieved for instance with stock index futures, such that its total risk is also 15.4%. Portfolio L is 60bp above the benchmark. This is a non-negligible fraction of the excess performance of 200bp. This illustrates the general point that part of the value added of the TEV portfolio P is fallacious.

2 Does Diversification Across Managers Pay?

Roll (1982) conjectured that this increase in risk could be mitigated by diversifying across active managers. Assuming the portfolio is equally invested in N managers, the total return is given by the return on the index plus the average of the active excess returns

$$R_P = \frac{1}{N}\sum_{i=1}^{N}(R_B + R_{\epsilon,i}) = R_B + \frac{1}{N}\sum_{i=1}^{N} R_{\epsilon,i} \tag{6}$$

The total portfolio variance can be derived from Equation (19). Assuming that all active excess positions have the same tracking risk and information ratio, we have

$$\sigma_P^2 = \sigma_B^2 + \frac{2}{N}\sum_{i=1}^{N} Cov[R_B, R_{\epsilon,i}] + \frac{1}{N^2}V[\sum_{i=1}^{N} R_{\epsilon,i}] = \sigma_B^2 + 2\sqrt{\frac{T}{d}}[\mu_B - \mu_{MV}] + \sigma^2[\frac{1}{N}\sum_{i=1}^{N} R_{\epsilon,i}] \tag{7}$$

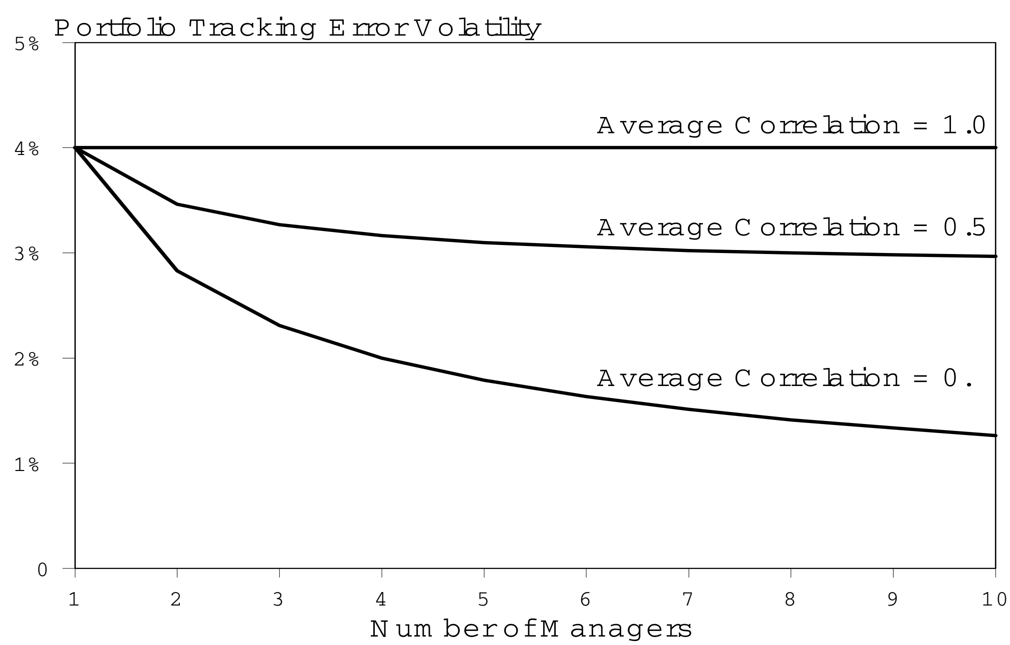
The second term represents the covariance between the index and the average portfolio deviation. This is positive and does not depend on the number of managers. The third term, in contrast, is affected by diversification. This represents the variance of the portfolio tracking error. Assuming all excess returns have the same correlation ρ with each other, it can be written as

$$\sigma[\frac{1}{N}\sum_{i=1}^{N} R_{\epsilon,i}] = \sigma_\epsilon\sqrt{\frac{1}{N} + (1 - \frac{1}{N})\rho}. \tag{8}$$

This term decreases with more managers or lower correlations.

Figure 3, however, shows that the rate of decrease is very small with realistic data. Assuming 10 managers and $\rho = 0.5$, for instance, the volatility of total tracking error only decreases from 4.0% to 3.0%. Thus, diversifying across managers is not likely to mitigate the inherent flaw in tracking error optimization.

FIGURE 3 Decrease in Tracking Risk with Multiple Managers



3 The Constant TEV Frontier

Now that we have shown the drawbacks of TEV optimization, the issue is whether additional constraints can be used to improve the performance of TEV-constrained portfolio. To do this, I now characterize the locus of points that correspond to a TEV constraint in the original MV space. The optimization can be written as

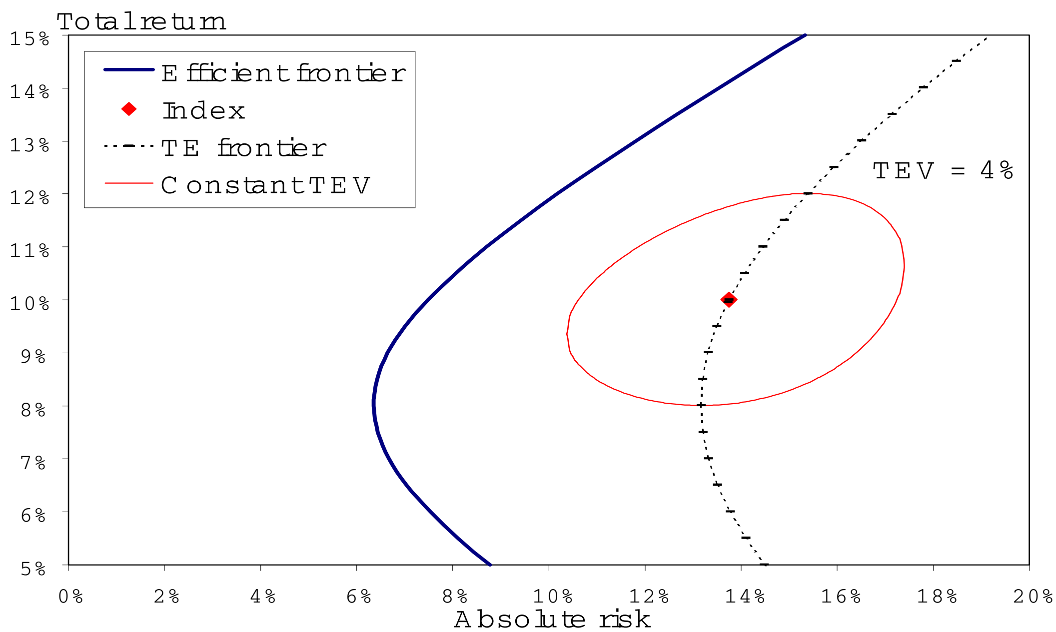
$$\begin{aligned} &\text{Max } x'E \\ &\text{s.t.} \qquad x'1 &= 0 \\ &\qquad x'Vx &= T \\ &\qquad (q+x)'V(q+x) &= \sigma_P^2 \end{aligned}$$

The first constraint sets the sum of portfolio deviations to zero. The second constraint sets the tracking error variance to a fixed amount T. Finally, the third constraint forces the total portfolio variance to be equal to a fixed value, σ_P^2. This number can be varied to trace out the constant TEV frontier. The solution is given in Appendix C.

For what follows, I define the quantities $\Delta_1 = \mu_B - \mu_{MV} \geq 0$ and $\Delta_2 = \sigma_B^2 - \sigma_{MV}^2 \geq 0$, which characterize the expected return and variance of the index in excess of that of the minimum variance portfolio. These quantities play a central role in the description of the TEV frontier. For this data set, $\Delta_1 = 2\%$ and $\Delta_2 = 0.0149$.

The relationship between expected return and variance for a fixed TEV turns out to be an ellipse. This is Equation (25) in Appendix C. The ellipse is somewhat distorted in the (σ, μ) space and is described in Figure 4.

FIGURE 4 Frontier with Constant Tracking Error Volatility

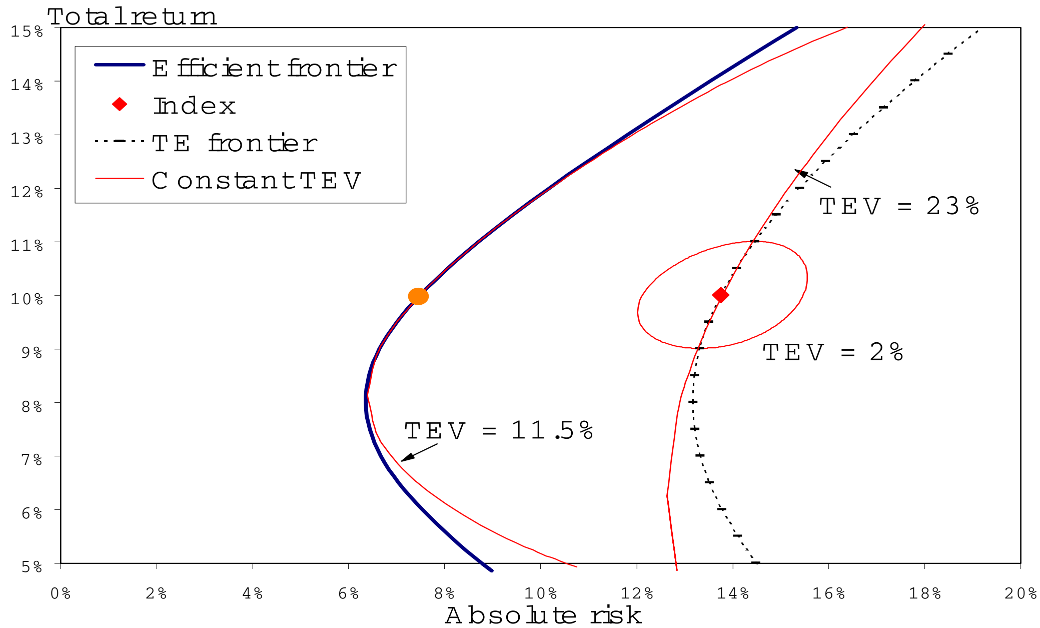


Next, Figure 5 shows the effect of changing TEV on this frontier. When σ_ϵ is zero, the ellipse collapses to a single point, the benchmark. As σ_ϵ increases, the size of the ellipse increases. The left side of the ellipse moves to the left and becomes tangent to the efficient set parabola at one point. First tangency occurs for $\sigma_\epsilon = \sqrt{\Delta_2 - \Delta_1^2/d} = 11.5\%$.

After that, there are two tangency points. As σ_ϵ increases, the ellipse moves to the right. For $\sigma_\epsilon = 2\sqrt{\Delta_2 - \Delta_1^2/d} = 23.0\%$, the ellipse passes through the index itself. All active portfolios with TEV constraints and positive excess returns must have greater risk than the index. These analytical results, proved in Appendix C, show that the tracking error volatility should be chosen very carefully. TEV values set too high make it impossible to maintain a level of risk similar to that of the benchmark.

Based on these results, we can now check whether the investor could induce the active manager to move closer to the efficient frontier by imposing additional constraints.

FIGURE 5 Constant TEV Frontiers for Various Steps



4 Moving Closer to the Efficient Frontier

4.1 Imposing a Risk-Return Trade-off

One solution would be for the investor to provide the manager with his or her risk-return tradeoff. The manager would then optimize the investor's utility subject to the TEV constraint. For instance, the problem can be set up as

$$\text{Max } U(\mu_P, \sigma_P) = \mu_P - \frac{1}{(2t)}\sigma_P^2 \tag{9}$$

where t is the investor's risk tolerance, subject to the TEV constraint.

The problem with this approach is that it is impractical to verify. Ex ante, the manager may not be willing to disclose expected returns. Ex post, realized returns are enormously noisy measures of expected returns. Instead, it is much easier to constrain the risk profile, either before or after the facts. No doubt this explains why managers are given tracking error constraints.

Armed with the equation for a constant TEV frontier, we can now explore the effectiveness of imposing additional restrictions. One such constraint, explored by Roll (1992), is to impose a beta of one. We can do more, however.

4.2 Imposing a Constraint on Total Risk

The investor could specify that the portfolio risk be equal to that of the index itself:

$$\sigma_P^2 = \sigma_B^2 \tag{10}$$

From Equation (3), this implies that $2q'Vx = -T$, or that the benchmark deviations must have a negative covariance with the index. Figure 5 shows that, when TEV is around 12%, such a constraint on absolute volatility can bring the portfolio much closer to the efficient set. Imposing an additional restriction on the manager, however, must decrease expected returns. The cost can be derived from Equation (35) in the Appendix. The issue is whether this restriction is really harmful.

The shape of the constant TEV frontier in Figure 4 suggests that this loss may not be large. This is because the top part of the ellipse is rather flat. The effects of a constraint on total volatility are illustrated in Table 2. The table reports, for various levels of σ_{MV} as well as of Δ_1, the drop in expected return and the associated reduction in volatility. The ratio of the drop in μ to that in σ can be viewed as the cost of the constraint.

When $\Delta_1 = 0\%$, that is $\mu_B = \mu_{MV}$, and $\sigma_{MV} = 8\%$, imposing a constraint on total volatility when the TEV is set at 4% leads to a loss of expected return of only 0.03%. This is in exchange for a risk reduction of 0.57%, for a ratio of 0.06. When $\Delta_1 = 2\%$, the loss of expected return is 0.29%, in exchange for a risk reduction of 1.65%, for a ratio of 0.18.

TABLE 2 Effect of Additional Constraint on Return and Risk

Active Portfolio Subject to a TEV Constraint

Constraint is that Portfolio Volatility Equals Index Volatility of 13.8%

	Tracking Error Volatility									
	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
$\Delta_1 = 0\%$	Drop in μ									
$\sigma_{MV} = 6\%$	0.00	0.00	-0.01	-0.03	-0.05	-0.09	-0.14	-0.21	-0.31	-0.43
$\sigma_{MV} = 8\%$	0.00	0.00	-0.01	-0.03	-0.06	-0.11	-0.18	-0.26	-0.38	-0.53
$\sigma_{MV} = 10\%$	0.00	-0.01	-0.02	-0.05	-0.09	-0.16	-0.25	-0.38	-0.54	-0.76
	Drop in σ									
	-0.04	-0.14	-0.32	-0.57	-0.88	-1.25	-1.68	-2.16	-2.68	-3.25
	Ratio: drop in μ/σ									
$\sigma_{MV} = 6\%$	0.01	0.02	0.03	0.05	0.06	0.07	0.09	0.10	0.11	0.13
$\sigma_{MV} = 8\%$	0.01	0.03	0.04	0.06	0.07	0.09	0.10	0.12	0.14	0.16
$\sigma_{MV} = 10\%$	0.02	0.04	0.06	0.08	0.10	0.12	0.15	0.17	0.20	0.23
$\Delta_1 = 1\%$	Drop in μ									
$\sigma_{MV} = 6\%$	-0.01	-0.03	-0.06	-0.10	-0.17	-0.25	-0.35	-0.47	-0.63	-0.81
$\sigma_{MV} = 8\%$	-0.01	-0.04	-0.07	-0.13	-0.20	-0.30	-0.43	-0.58	-0.77	-1.00
$\sigma_{MV} = 10\%$	-0.02	-0.05	-0.10	-0.18	-0.28	-0.42	-0.60	-0.82	-1.09	-1.42
	Drop in σ									
	-0.18	-0.43	-0.74	-1.12	-1.55	-2.03	-2.56	-3.13	-3.74	-4.39
	Ratio: drop in μ/σ									
$\sigma_{MV} = 6\%$	0.06	0.07	0.08	0.09	0.11	0.12	0.14	0.15	0.17	0.19
$\sigma_{MV} = 8\%$	0.07	0.08	0.10	0.11	0.13	0.15	0.17	0.19	0.21	0.23
$\sigma_{MV} = 10\%$	0.10	0.12	0.14	0.16	0.18	0.21	0.23	0.26	0.29	0.32
$\Delta_1 = 2\%$	Drop in μ									
$\sigma_{MV} = 6\%$	-0.03	-0.08	-0.15	-0.24	-0.35	-0.48	-0.64	-0.84	-1.06	-1.32
$\sigma_{MV} = 8\%$	-0.04	-0.10	-0.18	-0.29	-0.42	-0.59	-0.79	-1.02	-1.30	-1.62
$\sigma_{MV} = 10\%$	-0.06	-0.14	-0.26	-0.41	-0.60	-0.83	-1.11	-1.44	-1.83	-2.28
	Drop in σ									
	-0.32	-0.71	-1.15	-1.65	-2.19	-2.77	-3.40	-4.06	-4.74	-5.46
	Ratio: drop in μ/σ									
$\sigma_{MV} = 6\%$	0.10	0.12	0.13	0.14	0.16	0.17	0.19	0.21	0.22	0.24
$\sigma_{MV} = 8\%$	0.13	0.14	0.16	0.18	0.19	0.21	0.23	0.25	0.27	0.30
$\sigma_{MV} = 10\%$	0.18	0.20	0.23	0.25	0.27	0.30	0.33	0.35	0.38	0.42

NOTE: The table reports the drop in mean and volatility of the active portfolio when imposing an additional constraint that the total risk be equal to that of the benchmark. $\Delta_1 = \mu_B - \mu_{MV}$ is a measure of inefficiency of the benchmark, taken as the difference between its expected return and that of the global minimum-variance portfolio. σ_{MV} also measures the inefficiency of the benchmark, as receding values indicate that the benchmark is further from the efficient set.

These are very favorable return-to-risk ratios compared to an intrinsic information ratio (return-to-risk) of 0.50. Thus the cost of the additional constraint on total volatility is rather low.

We can also characterize conditions under which this constraint is most useful. This depends on the size of the tracking error constraint and the efficiency of the benchmark. First, the lower the TEV, the more helpful the constraint. Indeed the ratio of the drop in expected return to drop in volatility decreases as one moves from the right of the table to the left. Second, the less efficient the index, the more helpful the constraint. This corresponds to situations where the parameter Δ_1 is low or where σ_{MV} is low, ceteris paribus. In both cases, a less efficient index leads to a lower risk-adjusted cost for the constraint.

Hence, imposing a constraint on total risk constraint appears sensible precisely in situations where the benchmark is relatively inefficient. If the active manager is confident that he or she can add value, then the manager should easily accept an additional constraint on total portfolio risk.

4.3 Portfolio Positions

Next, I illustrate the effect of these constraints on portfolio positions. The results obtained so far depend only on the efficient set parameters and the characteristics of the benchmark. They hold for any number of assets. Table 3 shows how these numbers could be achieved with hypothetical expected returns for four global equity markets and a bond index.[9]

The information ratio of 0.5 is primarily driven by the dispersion in expected returns, shown in the second column. I chose high expected returns for German

[9] In practice, there can be substantial estimation error in expected returns when estimated from historical data. This is why we do not use historical data but instead adjust expected returns to achieve a "reasonable" information ratio. As Michaud (1989) has shown, the optimal portfolio is quite sensitive to errors in expected returns. When data are taken from historical observations, Jorion (1992) shows that the variability in the weights can be gauged from simulations based on the original sample. In contrast, the covariance matrix is more precisely estimated. Chan et al. (1999) showed that there is substantial predictability in the covariance matrix for optimization purposes.

and U.K. equities, moderate returns for U.S. equities, and low expected returns for Japanese equities. U.S. bonds are expected to return 8%. The next column shows the positions for the benchmark; these weights correspond to those in the global stock index in 2000. As before, the index is expected to return 10%.

The next two columns display positions in the usual TEV-constrained portfolio. To increase returns, the active manager increases the position in German and U.K. equities and decreases the position in Japanese equities, U.S. equities, and bonds. This increases the expected return by 2%. The total risk, unfortunately, also increases from 13.8% to 15.4%.

TABLE 3 Illustrative Positions

		Positions				
	Expected Return	Bench-mark	Portfolio: TEV-constrained		Portfolio: TEV-constrained Risk-constrained	
Asset	μ	q	x	$q+x$	x	$q+x$
German equities	14.7%	6.6%	10.5%	17.1%	9.8%	16.4%
Japanese equities	5.7%	17.5%	-10.6%	6.9%	-13.4%	4.1%
U.K. equities	14.7%	12.2%	17.5%	29.7%	16.3%	28.5%
US equities	9.8%	63.7%	-6.7%	57.0%	-16.8%	46.9%
US bonds	8.0%	0.0%	-10.7%	-10.7%	4.1%	4.1%
Total weight		100.0%	0.0%	100.0%	0.0%	100.0%
Expected Return		10.0%	2.0%	12.0%	1.8%	11.8%
Risk		13.8%	4.0%	15.4%	4.0%	13.8%

NOTE: The table reports expected returns and positions for three portfolios: the benchmark, the 4%-TEV constrained active portfolio, and the portfolio with an additional constraint that the total risk must equal that of the benchmark.

The last columns report positions for the TEV-constrained portfolio with an additional constraint on total risk. This portfolio has indeed a lower volatility, of 13.8%, which is equal to that of the benchmark. The most interesting aspect of the table is that this comes at a very low cost: The expected return is only marginally lower than before, at 11.8% instead of 12%. This is achieved by shorting more U.S. equities and moving the proceeds into U.S. bonds, which have low total risk.

Thus, adding a constraint on total risk preserves most of the benefits from active management. It remedies the inherent flaw in excess return optimization at a little cost in terms of average returns.

5 Conclusions

The common practice in the investment management industry is to control the risk of active managers by imposing a constraint on tracking risk. This setup, however, is seriously inefficient. When myopically focusing on excess returns, the active manager ignores the total risk of the portfolio. As a result, an optimization in excess returns that includes the benchmark assets will always increase the total portfolio risk relative to the benchmark.

This practice is reinforced by the widespread use of information ratios as performance measures. Because information ratios solely consider tracking error risk, they pay no attention to total portfolio risk.

This issue has major consequences for performance measurement: Part of the value added of active managers acting in this fashion is illusory, as it could be naively obtained by leveraging up the benchmark.

It is not clear why this problem, first pointed out by Roll in 1992, has not led to a de-emphasis on tracking error constraints and information ratios in the industry. If TEV constraints cannot be eliminated, perhaps the inefficiency of this approach can be mitigated by additional constraints. This is why this paper derives analytical solutions for the risk-return relationship of portfolios subject to a TEV constraint. This is shown to be an ellipse in the mean-variance plane. This analytical solution allows us to explore the effect of imposing additional constraints on the active manager.

The simplest constraint is to force the total portfolio volatility to be no greater than that of the benchmark. With the advent of forward-looking risk measures such as VAR, such constraint is easy to set up. I show that, due to the flat shape of this ellipse, adding such a constraint can substantially improve the performance of the active portfolio. This constraint is most beneficial in situations with low values of the admissible TEV, or when the benchmark is relatively inefficient.

In summary, the first prescription of this paper would be to discard TEV optimization and instead focus on total risk. There are indications indeed that pension plans with advanced risk management systems are now moving in this direction.[10] Otherwise, if TEV constraints are to be kept in place, an alternative recommendation would be to impose an additional constraint on total volatility. This paper provides the tools to do so.

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[10]This is the case, for instance, with ABP, the Dutch pension plan with $140 billion in assets, which currently ranks as the world's second largest pension fund. The plan assigns total risk limits to its active managers.

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Appendix A: The Mean-Variance Efficient Frontier

This appendix reviews the derivation of the conventional efficient frontier without a risk-free asset. Define G as the target expected return. The allocation problem involves a constrained minimization of the portfolio variance, over the weights q_P:

$$\begin{aligned} &\text{Min } q_P' V q_P \\ &\quad \text{s.t.} \quad q_P' 1 = 1 \\ &\qquad\quad\; q_P' E = G \end{aligned}$$

Following Merton (1972), the efficient set constants are defined as

$$a = E'V^{-1}E, \quad b = E'V^{-1}1, \quad c = 1'V^{-1}1, \quad d = a - b^2/c$$

The efficient frontier can be fully defined by two portfolios, one that minimizes the variance (MV) and the other that maximizes the return-to-risk ratio, also called the tangent (TG) portfolio to the efficient set, with weights

$$q_{MV} = V^{-1}1/c, \quad q_{TG} = V^{-1}E/b \tag{11}$$

The expected return and variance of the two portfolios are $E_{TG} = a/b, V_{TG} = a/b^2$, and $E_{MV} = b/c, V_{MV} = 1/c$. When the covariance matrix is positive definite, the constants a and c must be positive. In addition, the efficient set is meaningful when the expected return on the tangent portfolio is greater than that on the minimum variance portfolio, which implies that $d > 0$.

Taking the Lagrangian and setting the partial derivatives to zero, the allocations for any portfolio can be described as a linear combination of the two portfolios,

$$q_P = \left(\frac{a - bG}{d}\right) q_{MV} + \left(\frac{Gb - \frac{b^2}{c}}{d}\right) q_{TG} \tag{12}$$

Computing the variance and setting $G = \mu_P$, the efficient set is represented by

$$\sigma_P^2 = \frac{a}{dc} - \frac{2b}{dc}\mu_P + \frac{1}{d}\mu_P^2 = \frac{1}{d}(\mu_P - \frac{b}{c})^2 + \frac{1}{c} = \frac{1}{d}(\mu_P - \mu_{MV})^2 + \sigma_{MV}^2 \tag{13}$$

which represents a parabola in the (σ^2, μ) space or a hyperbola in the (σ, μ) space, with asymptotes having a slope of $\pm\sqrt{d}$. This slope represents the best return-to-risk ratio for this set of assets.

Appendix B: Tracking Error Frontier

This appendix derives the shape of the TE frontier in the *excess* mean-variance space, i.e. relative to a benchmark. We have to assume that the benchmark is not on the efficient set, otherwise there would be no rationale for active management. In addition, the expected return on the benchmark is assumed to be greater than, or equal to, that of the minimum variance portfolio: $\mu_B \geq \mu_{MV} = b/c$. If this was not satisfied, the benchmark would be grossly inefficient, as the investor could pick another index with the same risk but higher expected return.

Consider a maximization of the portfolio excess return over the deviations from the benchmark x:

$$\begin{aligned} &\text{Max } x'E \\ &\quad \text{s.t.} \quad x'1 &= 0 \\ &\quad\quad\quad x'Vx &= \sigma_\epsilon^2 = T \end{aligned}$$

Set up the Lagrangian as

$$L = x'E + \lambda_1(x'1 - 0) + 0.5\lambda_2(x'Vx - T) \tag{14}$$

Taking partial derivatives with respect to x and setting to zero, the solution is of the form

$$x = \frac{-1}{\lambda_2}V^{-1}[E + \lambda_1 1] \tag{15}$$

Selecting the values of the λs so that the two constraints are satisfied, we have

$$x = \pm\sqrt{\frac{T}{d}}V^{-1}[E - \frac{b}{c}1] \tag{16}$$

Note that the deviations x do not depend on the benchmark. This unexpected result is due to the fact that the portfolio manager only considers tracking error risk.

Solving now for the portfolio expected excess return, we have

$$\mu_\epsilon = \pm\sqrt{d}\ \sigma_\epsilon \tag{17}$$

where the upper part is a straight line in the tracking error space.

This can be translated back into the usual mean-variance space:

$$\mu_P = (q+x)'E = \mu_B \pm \sqrt{d}\sqrt{T} \tag{18}$$

$$\sigma_P^2 = (q+x)'V(q+x) = \sigma_B^2 \pm 2\sqrt{\frac{T}{d}}[\mu_B - \mu_{MV}] + T \tag{19}$$

Substituting for T, this represents a hyperbola in the (σ_P, μ_P) space with the same asymptotes as the conventional efficient frontier. When the benchmark is efficient, this collapses to the efficient frontier.

Appendix C: TEV Frontier in Absolute Return Space

This appendix derives the shape of the constant TEV frontier in the original mean-variance space. Define the quantities $\Delta_1 = \mu_B - b/c = \mu_B - \mu_{MV}$ and $\Delta_2 = \sigma_B^2 - 1/c = \sigma_B^2 - \sigma_{MV}^2$, which characterize the expected return and variance of the index in excess of that of the minimum variance portfolio. Δ_2 is always positive since σ_{MV}^2 is by definition the variance of the minimum-variance portfolio. Δ_1 should also be positive, as explained previously.

C.1: Derivation of the Frontier

Theorem 1 (Shape of the TEV-constrained frontier):
The constant TEV frontier is an ellipse in the (σ^2, μ) space, centered at μ_B and $\sigma_B^2 + T$. Defining the deviations from the center as $y = \sigma_P^2 - \sigma_B^2 - T$ and $z = \mu_p - \mu_B$, the constant TEV frontier is given by Equation (25).

Consider a maximization, or equivalently a minimization, over x:

Max $x'E$

$$\text{s.t.} \quad \begin{aligned} x'1 &= 0 \\ x'Vx &= T \\ (q+x)'V(q+x) &= \sigma_P^2 \end{aligned}$$

We set up the Lagrangian as

$$L = x'E + \lambda_1(x'1 - 0) + 0.5\lambda_2(x'Vx - T) + 0.5\lambda_3(x'Vx + 2q'Vx + q'Vq - \sigma_P^2) \tag{20}$$

Taking partial derivatives with respect to x and setting to zero, the solution is of the form

$$x = \frac{-1}{\lambda_2 + \lambda_3} V^{-1}[E + \lambda_1 1 + \lambda_3 V q] \tag{21}$$

We now select the values of the λs so that the three constraints are satisfied. We have

$$b + \lambda_1 c + \lambda_3 = 0$$

$$a + \lambda_1^2 c + \lambda_3^2 \sigma_B^2 + 2b\lambda_1 + 2\mu_B \lambda_3 + 2\lambda_1\lambda_3 = T(\lambda_2 + \lambda_3)^2$$

$$\mu_B + \lambda_1 + \lambda_3 \sigma_B^2 = (\sigma_P^2 - \sigma_B^2 - T)(\lambda_2 + \lambda_3)/2$$

Define $y = \sigma_P^2 - \sigma_B^2 - T$. Solving for the λs, we find

$$\lambda_3 = -\frac{\Delta_1}{\Delta_2} \pm \frac{y}{\Delta_2}\sqrt{\frac{(d\Delta_2 - \Delta_1^2)}{(4T\Delta_2 - y^2)}} \tag{22}$$

$$\lambda_1 = -\frac{(\lambda_3 + b)}{c} \tag{23}$$

$$\lambda_2 + \lambda_3 = \pm(-2)\sqrt{\frac{(d\Delta_2 - \Delta_1^2)}{(4T\Delta_2 - y^2)}} \tag{24}$$

Now define $z = \mu_P - \mu_B$. Replacing in (21), we compute $x'E$. The relationship between y and z can be derived as

$$\mathbf{d\ y^2 + 4\Delta_2 z^2 - 4\Delta_1 zy - 4T(d\Delta_2 - \Delta_1^2) = 0} \tag{25}$$

For a quadratic equation of the type $Ay^2 + Bz^2 + Cyz + F = 0$, this represents an ellipse when the term

$$AB - (1/4)C^2 = d\ (4\Delta_2) - (1/4)(-4\Delta_1)^2 = 4(d\Delta_2 - \Delta_1^2)$$

is strictly positive. This must be the case when the benchmark is within the efficient set. The efficient set equation (13) requires that $d\Delta_2 - \Delta_1^2 \geq 0$.

The main axis of the ellipse is not horizontal but instead has a positive slope when $\Delta_1 = \mu_B - \mu_{MV} > 0$. If the expected return on the benchmark happens to be equal to that of the minimum variance portfolio, the ellipse is horizontal.

C.2: Properties of the Frontier

We can further analyze the properties of this ellipse, which describes portfolios with constant tracking error volatility (TEV) in the mean-variance space.

Centering of Ellipse:

The vertical center of the ellipse is the expected return of the index, μ_B. The horizontal center of the ellipse is displaced to the right by the amount of tracking error variance, $\sigma_B^2 + T$. Thus increasing tracking error shifts the center of the ellipse to higher total risk regions.

Maximum and Minimum Expected Returns:

Since the maximum and minimum expected excess returns are obtained from the TEV frontier in excess return space, the absolute maximum and minimum expected returns on the constant TEV line are achieved at the intersection with the tracking error frontier. From Equation (18), this is

$$\mu_P = \mu_B \pm \sqrt{d\ T} \tag{26}$$

Faced with only a TEV constraint, the active manager will simply maximize the expected return for a given T. The problem is that this can substantially increase the total portfolio risk. At this point, the variance is

$$\sigma_P^2 = \sigma_B^2 + T + 2\Delta_1\sqrt{\frac{T}{d}} \tag{27}$$

Hence the active portfolio risk increases not only directly with the TEV, but also with the quantity $\Delta_1 = \mu_B - \mu_{MV}$. Increasing Δ_1 means, with a fixed σ_B^2, that the benchmark becomes more efficient. If so, active management must substantially increase portfolio risk.

Maximum and Minimum Variance:

With constant TEV, the absolute maximum and minimum values for the variance along the ellipse are given by

$$\sigma_P^2 = \sigma_B^2 + T \pm 2\sqrt{T(\sigma_B^2 - \sigma_{MV}^2)} \tag{28}$$

which do not depend on expected returns. Hence the width of the ellipse depends not only on the TEV, but also on the distance between the variance of the index and that of the global minimum variance portfolio.

C.3: Effect of Changing TEV

We now analyze the effect of changing TEV on these limits.

Theorem 2a (Minimum TEV for contact with efficient set):

When $\sigma_\epsilon = \sqrt{\Delta_2 - \Delta_1^2/d}$*, the constant TEV frontier achieves first contact with the efficient set; this occurs for a level of expected return equal to that of the benchmark.*

Figure 5 shows that, for large enough values of the TEV, portions of the ellipse touch the efficient set. We analyze the contact points between this ellipse and the efficient set parabola. Replace σ_p^2, and y, from (13) in (25). This gives

$$0 = d[\frac{1}{d}(z+\Delta_1)^2 + \frac{1}{c} - \sigma_B^2 - T]^2 + 2\Delta_2 z^2 - 4\Delta_1 z[\frac{1}{d}(z+\Delta_1)^2 + \frac{1}{c} - \sigma_B^2 - T] - 4T(d\Delta_2 - \Delta_1^2)$$

This is a quartic equation in z. After simplification, this gives

$$z^4 - 2z^2(dT - d\Delta_2 + \Delta_1^2) + (dT - d\Delta_2 + \Delta_1^2)^2 = (z^2 - k)^2 = 0 \tag{29}$$

This has a solution when $k = dT - d\Delta_2 + \Delta_1^2 \geq 0$, or when T is large enough. When there is no solution, the curves do not intersect. There is one contact point only when $k = 0$, or when the tracking error variance is

$$\sigma_\epsilon^2 = T_A = \Delta_2 - \Delta_1^2/d \tag{30}$$

at which point contact occurs for $z = 0$, or when $\mu_p = \mu_B$. In other words, first contact with the efficient set occurs on the horizontal from the index. In Figure 5, this is achieved for TEV=11.5% for our example. As T increases, there are two contact points, for which $z = \pm\sqrt{k}$.

Theorem 2b (TEV and minimum risk):

When $\sigma_\epsilon = \sqrt{\Delta_2}$*, the constant TEV frontier achieves a minimum level of risk equal to that of the global minimum-variance portfolio.*

Equation (28) can also be written as

$$\sigma_P^2 - \sigma_{MV}^2 = [\sqrt{T} \pm \sqrt{\Delta_2}\ ]^2 \tag{31}$$

The portfolio achieves minimum risk when $\sigma_P^2 = \sigma_{MV}^2$, or when

$$\sigma_\epsilon^2 = T_B = \Delta_2 \tag{32}$$

At this point, the lowest portfolio variance along the ellipse coincides with the global minimum variance portfolio. In our example, this is achieved for TEV=12.2%.

Theorem 2c (index outside the TEV frontier)

When $\sigma_\epsilon = 2\sqrt{\Delta_2 - \Delta_1^2/d}$, the constant TEV frontier passes through the benchmark itself. Above this value, the benchmark is no longer within the constant TEV frontier.

The ellipses passes through the benchmark position when, setting $y = -T$ and $z = 0$ in (25),

$$dT^2 - 4T(d\Delta_2 - \Delta_1^2) = 0$$

which implies

$$T_C = 4(\Delta_2 - \Delta_1^2/d) \tag{33}$$

In Figure 5, this is achieved for TEV=23.0% for our example. Beyond this point, all TEV-constained portfolios with positive excess return must have greater risk than the index.

Theorem 2d (benchmark risk and TEV frontier):

When $\sigma_\epsilon = 2\sqrt{\Delta_2}$, the constant TEV frontier achieves a minimum level of risk equal to that of the benchmark. Above this value, any constant TEV portfolio has greater risk than that of the benchmark.

Increasing T further moves the ellipse back to the right. In particular, when

$$\sigma_\epsilon^2 = T_D = 4\Delta_2, \tag{34}$$

we have $\sigma_P^2 = \sigma_B^2$. In our example, this is achieved for TEV=24.4%. Beyond this point, all TEV-constrained portfolios must have greater risk than the index.

C.4: Constraint on Total Risk

We now use the equation of the TEV ellipse to compute the expected return when the total risk is set to that of the index. Evaluating (25) at $y = \sigma_P^2 - \sigma_B^2 - T = -T$, we have

$$\mu_P - \mu_B = -T\frac{\Delta_1}{2\Delta_2} \pm \sqrt{T(d - \frac{\Delta_1^2}{\Delta_2})(1 - \frac{T}{4\Delta_2})} \tag{35}$$