The elasticity of interest rate volatility: Chan, Karolyi, Longstaff, and Sanders revisited

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This paper presents a careful reexamination of the results of Chan, Karolyi, Longstaff, and Sanders (CKLS) in *Journal of Finance*, **47** (1992), 1209–1227. By redefining the possible regime shift period in line with evidence from known policy changes and past empirical research, the present authors find evidence that contradicts the major results in their paper. The widely cited conclusion of their paper is that the elasticity of interest rate volatility, the coefficient linking interest rate volatility to interest rate levels, is 1.5. CKLS also concluded that there was no structural shift in the interest rate process after October 1979. When the structural shift period is defined to be temporary and coincident with the Federal Reserve Experiment of October 1979 through September 1982, it is found that there is strong evidence of a structural break. Furthermore, there is also evidence that, contrary to CKLS's claim, a moderately elastic interest rate process can capture the dependence of volatility on the level of interest rates, while highly elastic models cannot. In particular, this study finds support for the square-root Cox–Ingersoll–Ross process.

1. INTRODUCTION

Modeling and estimating volatility is of crucial importance in finance. Volatility is central to risk management, for instance in computing value-at-risk for portfolios over short horizons. Volatility is also central to the pricing and hedging of options and derivatives with option-like characteristics.

A number of interest rate models that are commonly used to price and hedge interest rate dependent securities begin with an assumed process for the instantaneous short rate of interest. These models differ most notably in the volatility structure assumed to govern interest rate movements. Casual observation suggests that the volatility of interest rate changes is not constant and may be related to levels of interest rates. The relation between the volatility of changes in interest rates and their level is termed the elasticity of interest rate volatility. Estimation of the elasticity of interest rate volatility is the subject of this paper.

Inferences drawn from model estimation and testing are intimately tied to the assumptions regarding parameter stability. And the question of changes in regime underlying an economic process is intimately tied to model choice. When a model is estimated over a finite time period, the assumption is necessarily made that the parameters of the model are constant during that period. This applies even to so-called time-varying parameter models since the time-variation in the coefficients of the basic model is modeled by means of constant-valued parameters tied to an underlying source of variation in some fixed way. Thus, if the economic process governing the time-variation has changed during the period used to estimate or test a model, the conclusions drawn from the estimates may be incorrect. In other words, tests for evidence of structural shifts are necessarily joint tests of the shift/no-shift hypotheses and a particular model of the process being studied.[1]

Chan, Karolyi, Longstaff, and Sanders (1992, hereafter CKLS) present a classic example of this dilemma. They compare a series of models for the short-term interest rate over the period 1964 through 1989 and conclude that an elasticity of volatility with respect to the level of interest rates of 1.5 is required to properly model the interest rate process. They also conclude that there is no evidence of a structural shift in the process after October 1979. This paper shows that by redefining the regime period, both conclusions are reversed. We demonstrate that, for the class of models CKLS studied, there is strong evidence of a regime shift during the Federal Reserve Experiment period of October 1979 through September 1982, and that, once this shift is controlled for in the estimation, high (1.5) elasticity models are rejected and some moderate (0.5 or 1.0) models are not. These results are robust to changes in the short-rate data used and to the treatment of outliers.

It is often the goal of macroeconomists to endogenize policy shifts so as to make structural changes part of their models. This approach, while laudable, is not always possible. Such a model may be intractable for specific applications. For instance, the diffusion and jump-diffusion models used in derivatives analysis are far richer than the simple vector autoregressive interest rate processes used in macroeconomic models of the economy that endogenize Federal Reserve policy actions.[2] The alternative is to approach the structural shift question from a purely statistical perspective. For instance, one can allow for two or more sets of parameter values to define the possible regimes, where the probability of being in a given regime is state-dependent.[3]

Two general approaches have been taken when estimating the interest rate behavior around the Federal Reserve Experiment period. Under the first approach, the parameters underlying the assumed interest rate process are allowed to change without ex ante identification of the period of the change. Hamilton (1988), Cai (1994), Ball and Torous (1995), and Gray (1996) model nonstationarities using a Markov-switching regime estimation technique, while Sanders and Unal (1988) and Duffee (1993) use Chow-type tests for structural changes around the Federal Reserve Experiment period.

Under the second approach, a model of interest rate behavior is suggested

[1] Suppose that the null hypothesis is that there was no structural change, and the test is conducted using Model A. If the null is not rejected, one cannot conclude with certainty that there was no structural shift. It may merely be the case that the test using Model A lacks the power to reject the null. Similarly, if the null is rejected it does not necessarily follow that there was a regime shift. It may be the case that Model A is too restrictive and that another model, Model B, may be better able to capture the variation in the data.

[2] See, for instance, Leeper, Sims, and Zha (1996).

[3] Gray (1996) and Dahlquist and Gray (1997) provide examples of this approach.

that is believed to be robust to the mean-variance changes observed during the Federal Reserve Experiment period. CKLS argue that a simple approximation to a diffusion process with high (but constant) elasticity of volatility adequately captures the dynamic nature of the interest rate process over the entire 1964–1989 period. Giovannini and Jorion (1989), Brenner, Harjes, and Kroner (1996), and Koedijk *et al.* (1997) examine hybrid models that allow volatility to depend on both the current level of interest rates (as in CKLS) and lagged squared shocks (as in a GARCH model), while Ait-Sahalia (1996) allows the persistence of interest rate shocks to be a nonlinear function of the level.

The time-series of short-term interest rates, shown in Figure 1 is suggestive of a change in the process during the late 1970s and early 1980s. Both the level and the volatility appear elevated.[4] This period coincides with what is known as the Federal Reserve Experiment. In October 1979, the Federal Reserve Board announced that it would focus more on monetary aggregates and less on interest rate levels as a means of combating historically high inflation. This was followed in early 1980 by the imposition of credit controls by President Carter. These produced a sharp drop in interest rates from February to May. However, credit controls proved counterproductive, were soon lifted, and interest rates rose back to their winter levels (see Figure 1) by the end of the summer. In the fall of 1982, the Fed reverted to a more 'balanced' approach. This paper investigates whether the change in policy is reflected in changes in the parameters of interest rate models and whether allowing for this change effects our conclusions concerning the elasticity of interest rate volatility.

The remainder of this paper is organized as follows. Section 2 discusses the class of single-factor diffusion models to be studied, the data and estimation method to be employed, and the hypothesis test used to test for structural shifts. Section 3 presents the empirical results; first replicating the CKLS results, and then showing the conclusions to be dependent on an incorrect definition of the regime shift period. The revised results are then shown to be robust to extension of the data sample to include recent observations and to the use of alternative and arguably better data sources. Section 4 summarizes and concludes.

2. METHODOLOGY

2.1 *Models Examined*

As in CKLS, this paper examines the broad class of single-factor diffusion processes defined by

$$\mathrm{d}r = \kappa(\mu - r)\,\mathrm{d}t + \sigma r^{\gamma}\,\mathrm{d}z.$$

While not exhaustive of single-factor models, it does include several of the most important diffusion processes used in practice, including the Cox, Ingersoll, and Ross (CIR) model and the Vasicek model, as well as others. The continuous

[4] There is also evidence of isolated outliers in 1974 and 1980, which, while not suggestive of changes in the underlying process, may affect the power of tests used to examine structural shifts.

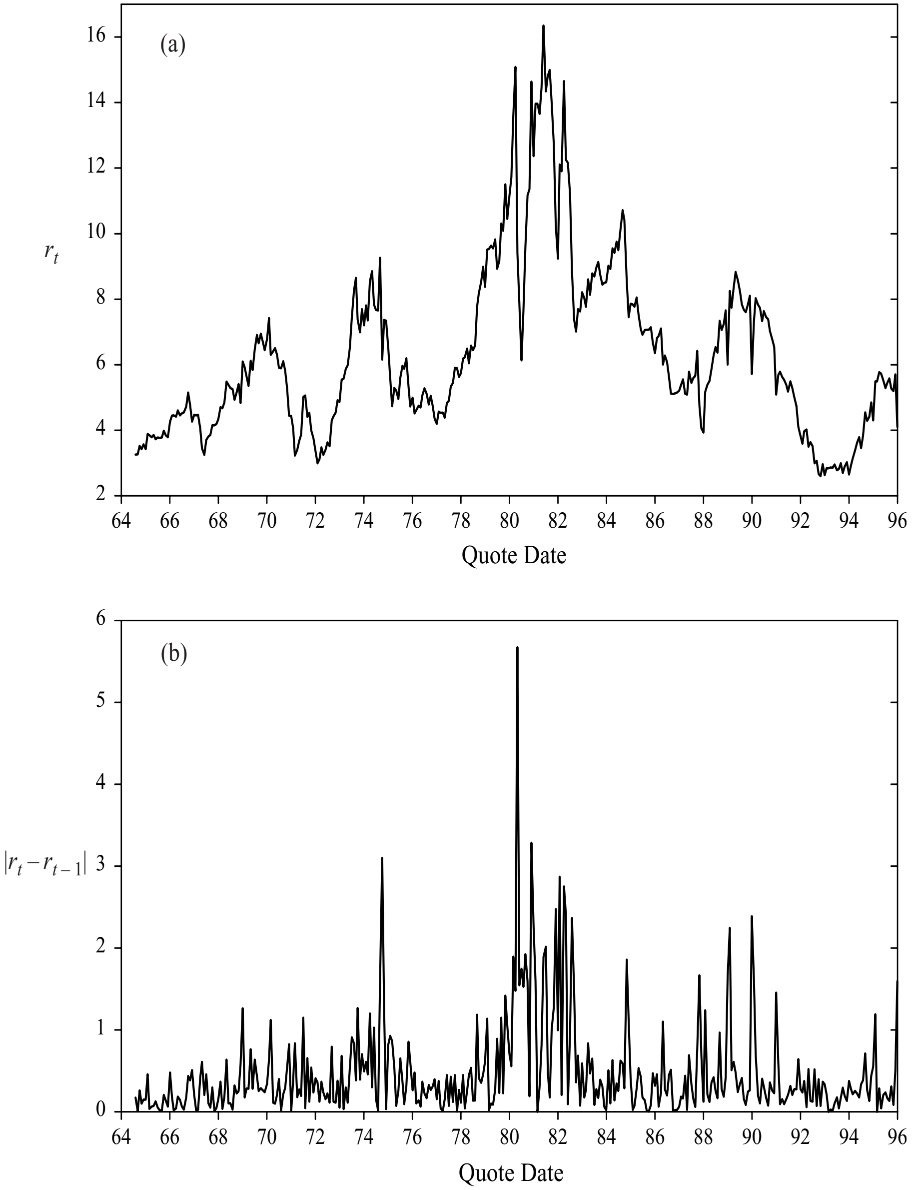

FIGURE 1. (a) The 1-month T-Bill rate; (b) absolute changes in the 1-month T-Bill rate.

time model in its general and restricted forms is estimated using the following discrete-time analog:

$$r_t - r_{t-1} = \alpha + \beta r_{t-1} + \epsilon_t, \tag{1}$$

$$\epsilon_t^2 = \sigma^2 r_{t-1}^{2\gamma} + \eta_t \tag{2}$$

The most general Base model is compared against eight special cases that impose restrictions on the values of α, β, σ, and γ.[5]

Model	Parameter restrictions				Degrees of freedom
	α	β	σ	γ	
Base	—	—	—	—	0
Merton	—	0.0	—	0.0	2
Vasicek	—	—	—	0.0	1
CIR-SR	—	—	—	0.5	1
Dothan	0.0	0.0	—	1.0	3
GBM	0.0	—	—	1.0	2
Brennan–Schwartz	—	—	—	1.0	1
CIR-VR	0.0	0.0	—	1.5	3
CEV	0.0	—	—	—	1

Dahlquist (1996) has pointed out that a number of these models may be unstationary.[6] This violates one of the few assumptions underlying the generalized method of moments (GMM) method to estimate these models, namely that the data be drawn from an ergodic process (see Hamilton 1988: p. 412). Where nonstationarity occurs, the standard errors derived from asymptotic results may be invalid. The constant-volatility Merton and Vasicek models permit the interest rate process to become negative (depending on the value of β in the latter case), a generally undesirable feature. Nonetheless, in the interest of comparability with the original CKLS study, we include these models and permit γ in the Base and CEV models to assume any positive value.

2.2 *Data*

For purposes of replicating and then reexamining the CKLS results, we use the 1-month Treasury Bill series taken from the 12-month Fama Treasury Bill Files included in the Center for Research in Security Prices (CRSP) monthly Government Bonds Files. We initially restrict our analysis to the June 1964 through December 1989 period to ensure comparability with CKLS.

This data series is not ideal for two reasons. First, the maturity of the nominal 1-month Treasury Bill in the 12-month file varies between 10 and 41 days. This is an artifact of the construction of the series. A more constant-maturity

[5] CKLS refer to the general model as the 'unrestricted model'. Designating this model as the 'Base model' avoids confusion later on when different forms of restrictions are applied in the regime shift tests. CIR-SR is the CIR square-root model, GBM is geometric Brownian motion, CIR-VR is the CIR variable rate model (used in their 1980 study of variable rate securities), and CEV is the constant elasticity of volatility model.

[6] The conditions for stationarity are developed by Broze, Scaillet, and Zakoian (1995). The condition $0 \leqslant \gamma \leqslant 1$ is a necessary condition. For $0 \leqslant \gamma < 1$, $(\beta + 1)^2 + \sigma^2 < 1$ is sufficient to guarantee stationarity, intuitively requiring mean reversion to be 'strong enough' to ensure the random innovations do not overcome the pull towards the mean. For $\gamma = 1$, $|\beta + 1| < 1$ is necessary and sufficient, intuitively ensuring that the process mean reverts ($\beta < 0$) but does not oscillate ($\beta < -1$).

1-month series is the 1-month Treasury Bill rate found in the 6-month Treasury Bill file, which varies in maturity from 21 to 39 days (23 to 35 for all but two months). Second, Duffee (1996) has shown that the 1-month Treasury Bill rate is an unreliable proxy for the short rate because it exhibits variations unrelated to other 1-month rates or other similar-maturity Treasury Bill rates.

In the second part of this study we test the robustness of our conclusions using several updated interest rates: the 1-month Treasury Bill rate taken from the CRSP 12-month Fama Treasury Bill File, the 3-month Treasury Bill rates taken from the CRSP 6-month Fama Treasury Bill File, and the 1- and 3-month Eurodollar rates taken from the Federal Reserve H15 Series. The Treasury series used in this part of the study cover the period June 1964 through December 1995; the Eurodollar rates cover the period January 1971 through February 1997.

2.3 *Model Estimation and Regime Shift Test*

The models are estimated using Hansen's (1982) generalized method of moments with the following orthogonality conditions:

$$\begin{aligned} g_t(\theta) &\equiv [\epsilon_t, \epsilon_t r_{t-1}, \eta_t, \eta_t r_{t-1}]^{\mathsf{T}}, \\ \mathsf{E}[g_t(\theta)] &= 0, \end{aligned} \tag{3}$$

where $\theta \equiv (\alpha, \beta, \sigma, \gamma)$ and ϵ_t and η_t are defined in equations (1) and (2). An estimate of θ can be obtained by choosing $\hat{\theta}$ to minimize the quadratic

$$J_T(\theta) = g_T(\theta)^{\mathsf{T}} W_T(\theta)\, g_T(\theta), \tag{4}$$

where

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} g_t(\theta)$$

is the sample average of the realizations of g_t, and $W_T(\theta)$ is a positive semidefinite weighting matrix. Under the null hypothesis, for large T, the sample average g_T will converge to zero when evaluated at the true values of θ.[7]

[7] The optimal weighting matrix is the inverse of the asymptotic covariance matrix

$$S = \lim_{T\to\infty} T \cdot \mathsf{E}\{[g_T(\theta)][g_T(\theta)]^{\mathsf{T}}\}.$$

Hansen (1982) shows that W_T can be consistently estimated by

$$W_T(\hat{\theta}) = \left[\frac{1}{T}\sum_{t-1}^{T} g_t(\hat{\theta})\, g_t(\hat{\theta})^{\mathsf{T}}\right]^{-1}$$

as long as $\hat{\theta}$ is a consistent estimate of θ and the $g_t(\theta)$ is serially uncorrelated. However, because estimating $W_T(\hat{\theta})$ requires an estimate of $\hat{\theta}$, an iterative procedure must be used. Following Hansen (1982), we first set $W_T(\theta)$ to the identity matrix and estimate $\hat{\theta}^0$ by minimizing $g_T(\theta)^{\mathsf{T}} I\, g_T(\theta)$. We then use $\hat{\theta}^0$ to compute $W_T(\hat{\theta}^0)$. This estimate of the weighting matrix is then used in the final iteration to estimate $\hat{\theta}$.

Ferson and Foerster (1994) and Hamilton (1994, p. 413) discuss the advantages of repeating the process of reestimating $W_T(\hat{\theta}^j)$ and then computing $\hat{\theta}^{j+1}$ until the $\hat{\theta}$'s converge. However, the single-iteration solution has the same asymptotic distribution, is commonly used, and produces solutions which tend to match those in CKLS (1992).

To test for regime shifts in the form of changes in the coefficients, the models are extended as follows:

$$r_t - r_{t-1} = (\alpha + \delta_1 D_t) + (\beta + \delta_2 D_t) r_{t-1} + \epsilon_t,$$
$$\epsilon_t^2 = (\sigma^2 + \delta_3 D_t) r_{t-1}^{2(\gamma+\delta_4 D_t)} + \eta_t,$$

where D_t is an indicator variable. The δ_i's measure the shift in the underlying parameters during the alternative regime period. In replicating the CKLS study, D_t is set to one beginning in October 1979 and zero before then. For the alternative temporary regime definition, D_t is set to one from October 1979 through September 1982, corresponding to the period of the Federal Reserve Experiment, and to zero both before and after that period. For example, for models in which α is estimated (e.g. CIR-SR), the parameter value is α outside of the regime shift period (prior to October 1979, or before October 1979 and after September 1982) and $\alpha + \delta_1$ during the shift. Where a model parameter is not freely estimated (e.g. γ in most models), the corresponding δ_i is fixed at zero.

To estimate the models, the following moment restrictions are used:

$$g_t(\theta) \equiv [\epsilon_t, \epsilon_t r_{t-1}, \epsilon_t D_t, \epsilon_t D_t r_{t-1}, \eta_t, \eta_t r_{t-1}, \eta_t D_t, \eta_t D_t r_{t-1}]^{\mathsf{T}},$$
$$\mathsf{E}[g_t(\theta)] = 0. \tag{5}$$

Formal tests for parameter shifts consist of running the various models, first in unrestricted form by estimating all relevant δ_i's and then estimating the restricted versions (δ_i's set to zero) using the weighting matrix from the unrestricted estimates (that is, not iterating to recompute W_T). Following Newey and West (1987), the test statistic is

$$R = T[J_T(\tilde{\theta}) - J_T(\hat{\theta})], \tag{6}$$

where $J_T(\hat{\theta})$ is the unrestricted model's objective function value and $J_T(\tilde{\theta})$ is the restricted model's objective function value where the unrestricted model's weighting matrix is used in both cases.[8] Newey and West (1987) point out that this procedure is analogous to a likelihood ratio test. Under the null hypothesis that the restrictions are not binding—in this case that there is no regime (parameter) shift and the unrestricted δ_i's are all statistically indistinguishable from zero—the test statistic R has a χ^2 distribution with degrees of freedom equal to the number of restrictions (number of δ_i's set to zero) under the alternative hypothesis.

The order of testing is important. If a structural shift is suspected, as it is in this case because of the Federal Reserve Experiment, prudence dictates that the issue of structural stability be addressed before drawing conclusions from

[8] Note that the 'unrestricted' model may be a restricted version of the general Base model, equations (1) and (2), and the 'restricted' version has the further restriction that all the δ_i's are zero. For example, the unrestricted Merton model would have 4 free parameters (α, σ^2, δ_1, and δ_3) while the restricted model has only two (α, σ^2).

models estimated using the entire sample period. The nature of the structural stability test used in CKLS (and here) takes as the null hypothesis that there was no structural shift in the parameter values. Thus, before proceeding to estimate and analyze full-sample results, it behooves us to first fail to reject the no-shift null hypothesis for plausible regime shifts (an unfortunately convoluted, if precise, double-negative formulation of the issue at hand). One can always use a more aggressive approach such as Hamilton switching regressions to search for any possible regime shift. But as a minimum, where the existing literature points to a regime shift, one should in good faith check the dates that have been so identified.

Because the other eight models are all nested within the general Base model, the test for a structural shift need only be applied to that model. If the test shows the existence of parameter shifts (rejects the no-shift null) for the general model, it is necessarily the case that a structural shift occurs for all restricted versions of that model. Restricting a model cannot make it fit the data better than the unrestricted model in which it is nested.[9] If, on the other hand, the test of the most general model fails to reject the no-shift null, we can then ask whether any restricted, and hence more parsimonious, versions of the model will also fit the data. *Ceteris paribus*, the interests of parsimony favor using the more restricted model in such a case. If, however, a restricted model fits the data in the unconstrained (regime-dummy coefficients free to vary) form and rejects the no-shift null (when the regime-dummy coefficients are forced to zero) while the general model shows no evidence of a regime shift, then the researcher is left to choose between the less parsimonious model with constant parameters and the more parsimonious model with changing parameters. How to make this choice is unclear since neither too many parameters nor too many parameter shifts are desirable.

3. EMPIRICAL RESULTS

3.1 *Replicating CKLS Results*

Table 1 presents the replication of CKLS basic model estimation results using the same data series, time period, and estimation methods.[10] The results of these

[9] Sampling variation may occasionally cause rejection of the no-shift null for the general model but fail to reject it for the restricted case. Such results are likely to be spurious and should be disregarded.

[10] Not having the same GAUSS code, we were unable to exactly match all coefficients. Where large differences occur, they appear to result from a difference in scaling rather than an inherently different solution. The salient features of levels of significance of the estimated parameters, model *p*-values and, most importantly, γ estimates, where applicable, are closely comparable. The notable exceptions are the value for σ^2 in the CEV model, 0.5207 (0.62) as compared with 0.1445 (4.57) in the Base model, and the significance of the σ^2 coefficient in the Base model, 0.77 versus 5.69.

TABLE 1. Interest rate models fitted to 1-month treasury bill rate.

Model	α	β	σ^2	γ	χ^2 (p-value)
Base	0.480 (1.85)	−0.592 (−1.55)	0.167* (5.69)	1.500* (5.95)	
Merton	0.005 (1.39)	0.0	0.0004* (6.31)	0.0	7.282 (0.026)
Vasicek	0.155 (0.76)	0.179 (0.50)	0.042* (6.22)	0.0	8.983 (0.003)
CIR-SR	0.186 (0.92)	−0.228 (−0.65)	0.074* (7.24)	0.5	6.444 (0.011)
Dothan	0.0	0.0	0.118* (7.89)	1.0	5.687 (0.128)
GBM	0.0	0.010 (1.48)	0.119* (7.94)	1.0	3.536 (0.171)
Brennan–Schwartz	0.267 (1.32)	−0.358 (−1.01)	0.119* (8.01)	1.0	2.644 (0.104)
CIR-VR	0.0	0.0	0.158* (8.01)	1.5	6.120 (0.106)
CEV	0.0	0.103 (1.77)	0.145* (4.57)	1.287* (4.22)	3.114 (0.078)

Notes:

- The Base model consists of

$$r_t - r_{t-1} = \alpha + \beta r_{t-1} + \epsilon_t,$$
$$\epsilon_t^2 = \sigma^2 r_{t-1}^{2\gamma} + \eta_t.$$

- Fixed parameters which define the different nested models are indicated by the absence of associated t-statistics.
- Data consist of monthly observations of the 1-month T-Bill rate for the period June 1964 through December 1989, a total of 306 observations after differencing.
- Numbers in parentheses below the parameter estimates are the asymptotic t-statistics; the asterisk indicates coefficients significant at the 5% level.
- $\chi^2 \equiv TJ_T$ is the value of the objective function J_T (see equation (3)), scaled to have an asymptotic χ^2 distribution. The corresponding p-values are shown in parentheses.

estimations lead to the following conclusions:

1. When γ is a free parameter, as in the Base and CEV models, the estimated value of γ exceeds unity.
2. Models which set γ to be less than one, and in particular the CIR-SR model, are rejected.

The result that $\gamma = 1.5$ is the most widely cited conclusion of the CKLS paper.

Table 2 presents the replication of the CKLS regime shift test, with the additional information that the unrestricted models' χ^2-statistics are shown as

TABLE 2. Tests for permanent regime shift at October 1979 in the 1-month T-Bill rate.

	Unrestricted models (parameter shift permitted)										Shift test	
	System parameters				Dummy parameters				χ^2-stat.		R-stat.	
Model	α	β	σ^2	γ	δ_1	δ_2	δ_3	δ_4	p-value	d.f.	p-value	d.f.
Base	0.017	−0.221	1.385	1.481*	0.061	−0.751	−0.208	−0.064			2.193	4
	(0.86)	(−0.53)	(0.47)	(4.03)	(1.18)	(−0.99)	(−0.06)	(−0.12)			(0.700)	
Merton	0.007	0.0	0.0002*	0.0	−0.002	0.0	0.001*	0.0	9.659	4	13.814	2
	(1.93)		(6.82)		(−0.21)		(3.66)		(0.047)		(0.001)	
Vasicek	−0.001	0.171	0.0002*	0.0	0.024	−0.417	0.001*	0.0	9.606	2	14.199	3
	(−0.07)	(0.45)	(6.82)		(0.53)	(−0.61)	(3.61)		(0.008)		(0.003)	
CIR-SR	0.002	0.108	0.004*	0.5	0.034	−0.512	0.007*	0.0	6.268	2	11.595	3
	(0.10)	(0.29)	(5.52)		(0.73)	(−0.76)	(3.27)		(0.044)		(0.009)	
Dothan	0.0	0.0	0.078*	1.0	0.0	0.0	0.066*	0.0	7.455	6	5.847	1
			(7.62)				(2.42)		(0.281)		(0.016)	
GBM	0.0	0.139	0.082*	1.0	0.0	−0.086	0.064*	0.0	3.696	4	6.579	2
		(1.89)	(7.71)			(−0.61)	(2.30)		(0.449)		(0.037)	
Brennan–Schwartz	0.008	−0.023	0.083*	1.0	0.048	−0.644	0.068*	0.0	2.048	2	6.518	3
	(0.44)	(−0.06)	(7.92)		(1.04)	(−0.96)	(2.41)		(0.359)		(0.089)	
CIR-VR	0.0	0.0	1.439*	1.5	0.0	0.0	0.157	0.0	7.137	6	0.226	1
			(7.97)				(0.48)		(0.308)		(0.634)	
CEV	0.0	0.131	0.491	1.305*	0.0	−0.080	−0.230	−0.188	3.193	2	1.636	3
		(1.77)	(0.43)	(3.27)		(−0.56)	(−0.18)	(−0.32)	(0.203)		(0.651)	

Notes:

- The unrestricted Base model consists of

$$r_t - r_{t-1} = (\alpha + \delta_1 D_t) + (\beta + \delta_2 D_t) r_{t-1} + \epsilon_t,$$
$$\epsilon_t^2 = (\sigma^2 + \delta_3 D_t) r_{t-1}^{2(\gamma + \delta_4 D_t)} + \eta_t,$$

  where D_t is an indicator variable set to 0 from June 1964 through September 1979 and 1 from October 1979 through December 1989. Restricted versions of the models have the δ_i's set to zero.

- Fixed parameters which define the different models have no associated t-statistics.
- Data consist of monthly observations of the 1-month T-Bill rate for the period June 1964 through December 1989, a total of 306 observations.
- Numbers in parentheses below the parameter estimates are the asymptotic t-statistics; the asterisk indicates coefficients significant at the 5% level.
- $\chi^2 \equiv TJ_T$ is the value of the objective function J_T (see equation (3)), scaled to have an asymptotic χ^2 distribution. The corresponding p-values are shown in parentheses.
- R-statistic is the test statistic for no-shift null hypothesis (see equation (4)). The corresponding p-values are shown in parentheses.

well. The regime definition tested is pre- versus post-October 1979. The same three models that are rejected when estimated over the entire sample period—Merton, Vasicek, and CIR-SR—are also rejected when their parameters are permitted to shift during the second part of the sample period. While CKLS do not discuss this point, models that are rejected in their unrestricted form should not then be used to test for significance of restrictions on the dummy parameter coefficients. Of the remaining six models, two reject the no-shift null hypothesis. These are the Dothan and GBM models, both of which set $\gamma = 1$ as well as restrictions on other parameters. This leaves the Base, Brennan–Schwartz, CIR-VR and CEV models, all of which fail to reject the data in their unrestricted forms (the Base model is exactly identified and thus cannot be rejected) and also fail to reject the no-shift null in the regime shift test. These models are all characterized by $\gamma \geqslant 1$. CKLS conclude that "...there is no evidence of a structural break in October 1979 for models that capture the dependence of the conditional variance on the level of the interest rate" (p. 1222). These results strengthen the conclusions drawn from the initial full-sample results.

Following the order of testing argued above, we might conclude that the Base, Brennan–Schwartz, CIR-VR, and CEV models fit the data without evidence of a regime shift in the parameter values. These are all characterized by high-γ values. Proceeding to full-sample analysis of these four models would therefore be appropriate. The only difference between these results and those obtained by looking directly at the full-sample results is that we would reject the other two high-γ models: the Dothan and GBM. This difference would not fundamentally undermine the conclusions drawn in CKLS: that there was no regime shift and that only high-γ models can fit the time series of short-term interest rates.

So, aside from observing methodological niceties, does this really matter?

3.2 *Reexamining CKLS Conclusions*

To see the effect of the methodological problem inherent in CKLS's procedure, one must first look at the choice of the regime period. The use of October 1979 to define a permanent structural break is curious given the widespread economic and statistical evidence that the structural break was temporary and ended in late 1982.[11] We therefore repeat the above analysis using the same methods, data,

[11] Hamilton (1988), Duffee (1993), Cai (1994), Koedijk *et al.* (1996), and Sanders and Unal (1988), all find evidence of a temporary regime shift roughly coincident with the Federal Reserve Experiment of October 1979 to fall 1982. Brenner, Harjes, and Kroner (1996) fit six models to two data series assuming parameter stationarity over the entire sample period. They then perform a series of misspecification tests, including a test of the variance-process residuals $\epsilon_t^2 - \sigma_t^2$ (but not the levels- process residuals ϵ_t) during the Federal Reserve Experiment period. Their results confirm the presence of Federal Reserve Experiment period-related misspecifications in some models. They do not discuss whether this test is more or less powerful than the more direct test for parameter shifts used in CKLS and this paper. Earlier studies, such as those of Huizinga and Miskin (1984) and Campbell (1987), did not check for an end to the Federal Reserve Experiment period, perhaps because they did not have sufficient post-1982 data. Ball and Torous (1995) find evidence of a single structural shift at October 1979 (contradicting the CKLS finding), but did not examine the possibility that the shift was temporary.

and sample period, but define the structural break period as running from October 1979 through September 1982. We model the period before and after the Federal Reserve Experiment period as a single regime. This is consistent with the results mentioned above and with the belief that the Federal Reserve reverted to its previous operating procedure after the experiment.[12] The results presented in Table 3 lead to a very different conclusion. The Base model rejects the no-shift null at the 5% level. Furthermore, the γ and β parameter estimates no longer violate stationarity conditions. We conclude that this is sufficient evidence of a structural break, and therefore sufficient to invalidate analysis of full-sample estimations.[13]

It is interesting to examine the CIR-VR model. Looking only at the shift test results, as CKLS do (they do not present the unrestricted model χ^2-statistics), one might conclude that this model is able to "...capture the dependence of the conditional variance on the level of the interest rate" (p.1222) because the no-shift null is not rejected in this instance. But this model is rejected in its unrestricted form and so cannot be used to form a judgement concerning the structural shift.

Given the evidence from the Base model that there was a structural shift from October 1979 through September 1982, we next examine whether any of the restricted variants of the Base model are able to fit the data. Two models, the CIR-SR and Brennan–Schwartz models, are not rejected in their unrestricted forms, when their parameter values are permitted to change. These models set $\gamma = 0.5$ and 1, respectively. The β estimates indicate both processes are stationary. If we look at the γ values in the unrestricted Base model, we find $\gamma = 0.95$ outside the temporary structural shift period and $\gamma = 0.33$ within the shift period, although this latter result is of doubtful statistical significance. With the structural shift period defined as October 1979 through September 1982, most of the high-γ models, including the CIR-VR which forces $\gamma = 1.5$, are rejected, as are the constant volatility Merton and Vasicek models, for which γ appears to be too low.

These results turn the CKLS results on their head. With the clear evidence of a structural shift, analysis of models based on estimates over the entire June 1964 through December 1989 period are misspecified. Furthermore, it is moderate-γ models, and notably the CIR-SR model, that fit the data when parameter shifts are permitted. We next test the robustness of this result to changes in short-rate series, period covered, and possible outliers.

3.3 *Robustness of Regime Shift Test*

We begin by identifying observations which may unduly influence our finding of a regime shift. For brevity we restrict the investigation to the Base model since it

[12] It is, of course, possible to apply the methodology used in this paper to test for three or more regimes if one wished. However, the concomitant increased parametrization will eventually lead to less reliable results.

[13] The failure of the Base model to reject the no-shift null for the permanent regime shift definition is immaterial. The assumption underlying full-sample analysis is that the structural stability holds for all possible regime definitions.

TABLE 3. Tests for temporary regime shift from October 1979 through September 1982 using the 1-month T-Bill rate.

	Unrestricted models (parameter shift permitted)										Shift test	
	System parameters				Dummy parameters				χ^2-stat.		R-stat.	
Model	α	β	σ^2	γ	δ_1	δ_2	δ_3	δ_4	p-value	d.f.	p-value	d.f.
Base	0.026*	−0.434*	0.071	0.948*	0.293	−2.346	−0.055	−0.616			10.665	4
	(2.39)	(−2.02)	(0.70)	(3.82)	(1.66)	(−1.53)	(−0.50)	(−0.87)			(0.031)	
Merton	0.005	0.0	0.0002*	0.0	−0.001	0.0	0.003*	0.0	10.890	4	21.448	2
	(1.77)		(7.43)		(−0.03)		(4.63)		(0.028)		(0.000)	
Vasicek	0.010	−0.093	0.0002*	0.0	0.295	−2.483	0.003*	0.0	8.387	2	24.489	3
	(1.04)	(−0.51)	(7.51)		(1.65)	(−1.63)	(4.14)		(0.015)		(0.000)	
CIR-SR	0.018	−0.245	0.005*	0.5	0.304	−2.594	0.028*	0.0	2.649	2	22.372	3
	(1.81)	(−1.34)	(8.21)		(1.69)	(−1.67)	(3.85)		(0.266)		(0.000)	
Dothan	0.0	0.0	0.086*	1.0	0.0	0.0	0.134*	0.0	12.779	6	6.207	1
			(7.90)				(2.49)		(0.047)		(0.013)	
GBM	0.0	0.045	0.086*	1.0	0.0	−0.228	0.139*	0.0	11.423	4	7.309	2
		(0.79)	(7.85)			(−0.76)	(2.61)		(0.022)		(0.026)	
Brennan–Schwartz	0.027*	−0.456*	0.095*	1.0	0.298	−2.550	0.186*	0.0	0.947	2	9.497	3
	(2.70)	(−2.38)	(8.34)		(1.66)	(−1.65)	(2.56)		(0.623)		(0.023)	
CIR-VR	0.0	0.0	1.288*	1.5	0.0	0.0	0.544	0.0	20.793	6	1.368	1
			(7.44)				(1.17)		(0.002)		(0.242)	
CEV	0.0	0.062	0.014	0.683*	0.0	−0.130	−0.008	−0.505	8.455	2	10.142	3
		(1.06)	(0.59)	(2.36)		(−0.41)	(−0.24)	(−0.63)	(0.015)		(0.017)	

Notes:

- The unrestricted Base model consists of

$$r_t - r_{t-1} = (\alpha + \delta_1 D_t) + (\beta + \delta_2 D_t) r_{t-1} + \epsilon_t,$$
$$\epsilon_t^2 = (\sigma^2 + \delta_3 D_t) r_{t-1}^{2(\gamma + \delta_4 D_t)} + \eta_t,$$

  where D_t is an indicator variable set to 1 from October 1979 through September 1982 and to 0 otherwise. Restricted versions of the models have the δ_i's set to zero.
- Data consist of monthly observations of the 1-month T-Bill rate for the period June 1964 through December 1989, a total of 306 observations.
- Numbers in parentheses below the parameter estimates are the asymptotic t-statistics; the asterisk indicates coefficients significant at the 5% level.
- Fixed parameters which define the different models are indicated by the absence of associated t-statistics.
- $\chi^2 \equiv TJ_T$ is the value of the objective function J_T (see equation (3)) scaled to have an asymptotic χ^2 distribution. The corresponding p-values are shown in parentheses.
- R-statistic is the test statistic for no-shift null hypothesis (see equation (4)). The corresponding p-values are shown in parentheses.

is this model which establishes the existence of the structural shift. Again, because our conclusions are critical of the original CKLS findings, we apply the test to their data set.

One method of identifying potential influential outliers is cross-validation. Cross-validation involves dropping an observation from the underlying data set and then reestimating the model and recording the parameter estimates. The dropped observation is then replaced and the next observation taken out. The process is repeated, producing a time-series of parameter estimates corresponding to the time-series of omitted observations. The four cross-validated parameters for the Base model, using the full sample period, are plotted in Figure 2 by the solid lines. Two conclusions may be drawn. First, there is near perfect positive correlation between the parameter estimates for σ and γ and negative correlation between the parameter estimates for α and β. This strong correlation suggests that some caution is needed when interpreting the values of individual coefficients, for instance γ.

Second, a single observation, March 1980, accounts for much of the high-γ conclusion reached in the original CKLS paper. The March 1980 drop in interest rates from 15.1 to 9.4% was almost 10 times the monthly standard deviation of interest rate changes over the sample period.

Dropping that single observation produces an estimate of $\gamma = 1.23$, which is not statistically different from unity. Repeating the cross-validation process for the remaining observations, with March 1980 permanently excluded, produces the dotted line in Figure 2. There is no evidence in this series of an additional

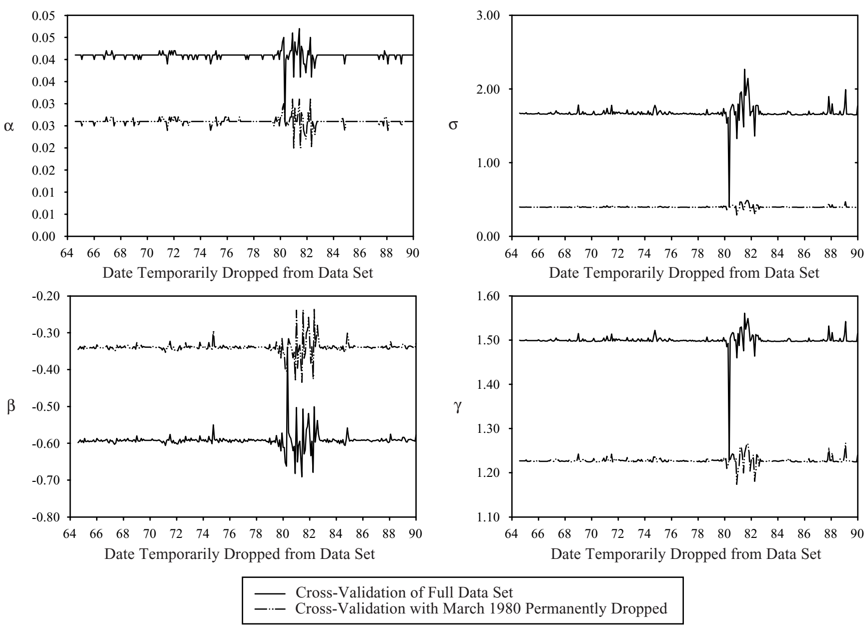

FIGURE 2. CKLS model cross-validation results.

observation likely to seriously impact the parameter estimates and hence our tests for a regime shift.

To examine whether our earlier conclusion that there was a temporary regime shift might be due to this single March 1980 observation, we repeat the regime shift test for the Base model with and without the March 1980 observation included in the sample. The results are presented in Table 4. In this case we present the restricted (full sample) as well as unrestricted model (structural shift) parameter estimates. These are for comparison with the original CKLS full-sample results only, but are statistically meaningless unless the no-shift null is not rejected. With the potentially anomalous March 1980 observation included, we reject the no-shift null as before. The (essentially meaningless) restricted model point estimate of γ is 1.22. This differs from the estimate in Table 1 because equation (5) was used to define the orthogonality conditions instead of equation (3). Furthermore, the Base model estimated using equation (3) is exactly identified and so cannot be rejected. When estimated in restricted form, using equation (5), the identical model is overidentified and thus capable of being rejected, which it is. When the March 1980 observation is dropped from the sample, the same qualitative results obtain: the restricted model is rejected, the no-shift null hypothesis is rejected, and the estimate of γ is 0.95 outside the Federal Reserve Experiment period and -0.223 (insignificantly different from zero) during the experiment. We conclude, therefore, that our earlier results are not due solely to the outlier.

The second source of potential problem in our analysis is the data set used. Criticisms of the 1-month T-Bill rate drawn from the 12-month T-Bill file were noted above. We therefore repeat our analysis using several alternative data sets:

- the same 1-month Treasury Bill series updated through December 1995;
- the 3-month Treasury Bill series which Duffee (1996) found to be free of the idiosyncratic variation found in the 1-month series;
- the 1-month Eurodollar series which Duffee recommended as an alternative to the 1-month Treasury Bill series; and
- the 3-month Eurodollar series (to compare with the 3-month T-Bill rate).

For each of these series, the above tests for a temporary October 1979 through September 1982 regime shift were repeated both with and without the suspect March 1980 observation. Then, conditional on evidence of a shift being found for the Base model, the restricted variants were tested to see if they could fit the data. The temporary regime shift test results for the 1-month Eurodollar rate are presented in Table 5. Again the presence of a structural break is established for the Base model. Tests of the CIR-SR, Brennan–Schwartz, and in this case the Dothan models are not rejected in their unrestricted form. However, when the regime dummies are forced to zero, these same models reject the no-shift null. Again, no model which fits the data fails to reject the no-regime null. Fixed-γ models which fit the data have $\gamma = 0.5$ or 1. Where γ is free to vary

TABLE 4. Another test for temporary regime shift from October 1979 through September 1982 using the 1-month T-Bill rate.

System parameters				Dummy parameters				χ^2-stat.		Shift test	
										R-stat.	
α	β	σ^2	γ	δ_1	δ_2	δ_3	δ_4	p-value	d.f.	p-value	d.f.
All observations											
Unrestricted Base model (parameter shift permitted)											
0.026*	−0.434*	0.071	0.948*	0.293	−2.346	−0.055	−0.616			10.665	4
(2.39)	(−2.02)	(0.70)	(3.82)	(1.66)	(−1.53)	(−0.50)	(−0.87)			(0.031)	
Restricted Base model (parameter shift not permitted)											
0.030*	−0.519*	0.367	1.229*					10.665	4		
(2.99)	(−2.66)	(1.03)	(7.01)					(0.031)			
March 1980 dropped											
Unrestricted Base model (parameter shift permitted)											
0.026*	−0.434*	0.071	0.948*	0.203	−1.434	−0.070	−1.171*			12.218	4
(2.39)	(−2.02)	(0.70)	(3.82)	(1.32)	(−1.12)	(−0.69)	(−2.41)			(0.016)	
Restricted Base model (parameter shift not permitted)											
0.026*	−0.416*	0.190	1.114*					12.218	4		
(2.70)	(−2.32)	(1.21)	(7.44)					(0.016)			

Notes:

- The unrestricted Base model consists of

$$r_t - r_{t-1} = (\alpha + \delta_1 D_t) + (\beta + \delta_2 D_t) r_{t-1} + \epsilon_t,$$

$$\epsilon_t^2 = (\sigma^2 + \delta_3 D_t) r_{t-1}^{2(\gamma + \delta_4 D_t)} + \eta_t,$$

  where D_t is an indicator variable set to 1 from October 1979 through September 1982 and to 0 otherwise. Restricted versions of the models have the δ_i's set to zero.

- Data consist of monthly observations of the 1-month T-Bill rate for the period June 1964 through December 1989, a total of 306 observations.

- Numbers in parentheses below the parameter estimates are the asymptotic t-statistics; the asterisk indicates coefficients significant at the 5% level.

- $\chi^2 \equiv TJ_T$ is the value of the objective function J_T (see equation (3)) scaled to have an asymptotic χ^2 distribution. The corresponding p-values are shown in parentheses.

- R-statistic is the test statistic for no-shift null hypothesis (see equation (4)). The corresponding p-values are shown in parentheses.

TABLE 5. Tests for temporary regime shift from October 1979 through September 1982 using the 1-month Eurodollar rate.

	Unrestricted models (parameter shift permitted)										Shift test	
	System parameters				Dummy parameters				χ^2-stat.		R-stat.	
Model	α	β	σ^2	γ	δ_1	δ_2	δ_3	δ_4	p-value	d.f.	p-value	d.f.
Base	0.022	-0.310	0.028	0.830*	0.402*	-2.566*	0.063	-0.032			13.554	4
	(1.88)	(-1.67)	(0.75)	(3.23)	(2.53)	(-2.18)	(0.23)	(-0.04)			(0.009)	
Merton	0.006	0.0	0.0003*	0.0	0.032	0.0	0.003*	0.0	21.208	4	10.015	2
	(1.77)		(5.24)		(0.87)		(2.77)		(0.000)		(0.007)	
Vasicek	0.023*	-0.269	0.0003*	0.0	0.351*	-2.191	0.003*	0.0	15.332	2	16.147	3
	(1.96)	(-1.44)	(5.43)		(2.24)	(-1.93)	(2.97)		(0.001)		(0.001)	
CIR-SR	0.025*	-0.327	0.005*	0.5	0.384*	-2.416*	0.021*	0.0	2.991	2	16.325	3
	(2.19)	(-1.76)	(6.55)		(2.39)	(-2.07)	(2.99)		(0.224)		(0.001)	
Dothan	0.0	0.0	0.062*	1.0	0.0	0.0	0.108*	0.0	10.963	6	5.238	1
			(5.94)				(2.29)		(0.090)		(0.022)	
GBM	0.0	0.023	0.062*	1.0	0.0	0.134	0.102*	0.0	10.384	4	5.166	2
		(0.42)	(5.93)			(0.49)	(2.15)		(0.034)		(0.076)	
Brennan–Schwartz	0.020	-0.285	0.065*	1.0	0.415*	-2.707*	0.146*	0.0	0.603	2	13.555	3
	(1.76)	(-1.56)	(6.21)		(2.60)	(-2.35)	(2.63)		(0.740)		(0.004)	
CIR-VR	0.0	0.0	0.734*	1.5	0.0	0.0	0.484	0.0	12.875	6	1.692	1
			(5.27)				(1.30)		(0.045)		(0.193)	
CEV	0.0	0.026	0.057	0.980*	0.0	0.121	-0.006	-0.264	9.793	2	4.748	3
		(0.47)	(0.70)	(3.53)		(0.44)	(-0.03)	(-0.27)	(0.008)		(0.191)	

Notes:

- The unrestricted Base model consists of

$$r_t - r_{t-1} = (\alpha + \delta_1 D_t) + (\beta + \delta_2 D_t) r_{t-1} + \epsilon_t,$$
$$\epsilon_t^2 = (\sigma^2 + \delta_3 D_t) r_{t-1}^{2(\gamma + \delta_4 D_t)} + \eta_t,$$

  where D_t is an indicator variable set to 1 from October 1979 through September 1982 and to 0 otherwise. Restricted versions of the models have the δ_i's set to zero.
- Fixed parameters which define the different models have no associated t-statistics.
- Data consist of monthly observations of the 1-month Eurodollar ratefor the period January 1971 through February 1997, a total of 313 observations.
- Numbers in parentheses below the parameter estimates are the asymptotic t-statistics; the asterisk indicates coefficients significant at the 5% level.
- $\chi^2 \equiv T J_T$ is the value of the objective function J_T (see equation (3)) scaled to have an asymptotic χ^2 distribution. The corresponding p-values are shown in parentheses.
- R-statistic is the test statistic for no-shift null hypothesis (see equation (4)). The corresponding p-values are shown in parentheses.

TABLE 6. Tests for temporary regime shift from October 1979 through September 1982 using various data sets.

Model	No observations dropped: Unrestricted model test	No observations dropped: 'No-shift' test	March 1980 dropped: Unrestricted model test	March 1980 dropped: 'No-shift' test
1-Month Treasury Bill from 12-month Fama File (Jun '64 through Dec '95)				
Base	N/A	Rejected	N/A	Rejected
Merton	Rejected	N/A	Rejected	N/A
Vasicek	Rejected	N/A	Rejected	N/A
CIR-SR	OK	Rejected	OK	Rejected
Dothan	Rejected	N/A	Rejected	N/A
GBM	Rejected	N/A	Rejected	N/A
Brennan–Schwartz	OK	Rejected	Rejected	N/A
CIR-VR	Rejected	N/A	Rejected	N/A
CEV	Rejected	N/A	Rejected	N/A
3-Month Treasury Bill from 6-month Fama File (Jun '64 through Dec '95)				
Base	N/A	Rejected	N/A	Rejected
Merton	Rejected	N/A	Rejected	N/A
Vasicek	Rejected	N/A	Rejected	N/A
CIR-SR	OK	Rejected	Rejected	N/A
Dothan	Rejected	N/A	Rejected	N/A
GBM	Rejected	N/A	Rejected	N/A
Brennan–Schwartz	OK	Rejected	Rejected	N/A
CIR-VR	Rejected	N/A	Rejected	N/A
CEV	Rejected	N/A	Rejected	N/A
1-Month Eurodollar rate from monthly H15 series (Jan '71 through Feb '97)				
Base	N/A	Rejected	N/A	Rejected
Merton	Rejected	N/A	Rejected	N/A
Vasicek	Rejected	N/A	Rejected	N/A
CIR-SR	OK	Rejected	OK	Rejected
Dothan	OK	Rejected	OK	Rejected
GBM	Rejected	N/A	Rejected	N/A
Brennan–Schwartz	OK	Rejected	OK	Rejected
CIR-VR	Rejected	N/A	Rejected	N/A
CEV	Rejected	N/A	Rejected	N/A
3-Month Eurodollar rate from monthly H15 series (Jan '71 through Feb '97)				
Base	N/A	Rejected	N/A	Rejected
Merton	Rejected	N/A	Rejected	N/A
Vasicek	Rejected	N/A	Rejected	N/A
CIR-SR	Rejected	N/A	Rejected	N/A
Dothan	OK	Rejected	OK	Rejected
GBM	Rejected	N/A	Rejected	N/A
Brennan–Schwartz	OK	Rejected	OK	Rejected
CIR-VR	OK	OK	OK	OK
CEV	Rejected	N/A	Rejected	N/A

Notes:

- The unrestricted model (parameters free to shift during alternative regime) is estimated first. Since the Base model is exactly identified, the test has no meaning (N/A) in this case.
- If the model is not rejected (OK), the no-shift hypothesis test is performed. If the unrestricted model is rejected, the no-shift test is inappropriate (N/A).

we obtain estimates close to unity outside the Federal Reserve Experiment period and below unity during the experiment.

Table 6 presents summaries of the temporary regime shift tests applied to all four data sets. The first result noted above, that there was a regime shift, is supported by all four data series. Dropping the March 1980 observation does not change this result. Thus we may conclude that the presence of the shift is robust, at least to models within the class studied by CKLS. The second result, that moderate-γ models also fit the data and that high- and low-γ models do not, is generally, but not invariably, supported. In all cases the constant-volatility Merton and Vasicek models are rejected. Similarly, the CEV model is always rejected. For the two 1-month interest rate series, the CIR-SR model is not rejected whether or not March 1980 is dropped, but for the 3-month Treasury Bill series it is rejected if that observation is omitted. For the 3-month Eurodollar series, the CIR-SR model is always rejected. The Brennan–Schwartz model is not rejected for the two Eurodollar series, and for the two Treasury series is only rejected if the March 1980 observation is omitted. The Dothan model is not rejected for the two Eurodollar series and is always rejected for the Treasury series. One anomalous result is the nonrejection of the CIR-VR model and the nonrejection of the associated no-shift null when using the 3-month Eurodollar rate. Since the existence of the shift is firmly established by the more general Base model within which the CIR-VR model is nested, and since this result is not repeated on other data sets, we are inclined to discount it.

4. SUMMARY AND CONCLUSIONS

This paper has argued that model selection and parameter stability are closely linked. We have shown this connection in a careful re-examination of the Chan, Karolyi, Longstaff, and Sanders (1992) paper. For instance, a widely cited conclusion of their paper is that the elasticity of interest rate volatility parameter γ is 1.5. This result depends crucially on a misspecification of the period during which the interest rate process changed in response to a temporary change in how the Federal Reserve implemented monetary policy. Defining the structural shift as coincident with this policy shift, we find that there is strong evidence in the interest rate data of a structural break. This result is robust to both changes in short-rate series and to the most influential outlier. Furthermore, we find evidence that, contrary to CKLS's claim, a moderate-γ interest rate process can capture the dependence of volatility on the level of interest rates while high-γ models cannot. This result is reasonably robust to changes in the short rate used and the treatment of outliers. In particular, this study finds support for the square-root CIR process, at least among the class of single-factor diffusion processes and for 1-month interest rates. There is no support for constant volatility models in either CKLS or this study.

These conclusions apply only to the class of models studied in this paper. It may well be the case that when other interest rate models, such as ARCH processes or jump diffusions, are applied, the apparent structural break will be found not to be significant. So long as this result can be reasonably shown to result from the superior structure of the model and not simply a comparative lack of power, we would be inclined to prefer such a process (if reasonably parsimonious) to one that required a relatively ad hoc, even if economically well founded, parameter shift to fit the data. In the meantime, users of diffusion processes, and probably other models as well, need to be aware of the potential for incorrect inferences when their estimation period spans this crucial period. Simple failure to reject a model estimated over a period spanning the well-documented Federal Reserve Experiment regime period is not sufficient to draw reliable conclusions. This was shown by CKLS's failure to reject the Dothan, GBM, CIR-VR, and CEV models over the entire period; all of which were rejected when the parameter shift was allowed for. Finally, modelers of interest rate processes can take comfort from knowing that the required link between interest rate volatility and levels is not so high as to produce troubling econometric and economic questions regarding stationarity.

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